Constructing Parametric Triangular Patches with Boundary Conditions ⋆

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Abstract

The problem of constructing a parametric triangular patch to smoothly connect three surface patches is studied. Usually, these surface patches are defined on different parameter spaces. Therefore, it is necessary to define interpolation conditions, with values from the given surface patches, on the boundary of the triangular patch that can ensure smooth transition between different parameter spaces. In this paper we present a new method to define boundary conditions. Boundary conditions defined by the new method have the same parameter space if the three given surface patches can be converted into the same form through parameter transformation. Consequently, any of the classic methods for constructing functional triangular patches can be used directly to construct a parametric triangular patch to connect given surface patches with G^1 continuity. The resulting parametric triangular patch preserves precision of the applied classic method.

Keywords: Triangular patch, Parametric interpolation, Determination of interpolation conditions

1 Introduction

Construction of surfaces plays an important role in computer aided geometric design (CAGD), free-form surface modeling and computer graphics (CG). To make the process of constructing complex surfaces simple, piecewise techniques are frequently used, with four-sided and triangular patches being the most popular choices. This paper studies the problem of boundary condition determination in the process of constructing parametric triangular patches to smoothly connect three given surface patches.

A curved triangular patch that interpolates the boundary interpolation conditions was first proposed by Barnhill, Birkhoff and Gordon [1]. The triangular patch is constructed using the *Boolean sum scheme*. Gregory [2] used the *convex combination method* to construct a triangular patch. The triangular patch is formed by the convex combination of three interpolation operators,

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each of which satisfies the interpolation conditions on two sides of a triangle. The idea [2] was further extended in papers [3, 4]. Nielson [5] presented a *side-vertex method* to construct a curved triangular patch using combination of three interpolation operators, each satisfying the given boundary conditions at a vertex and its opposite side. Hagen [6] extended Nielson's approach to construct *geometric patches*. These results have been generalized to triangular patches with first and second order geometric continuity [7, 8]. The problem of constructing non-four-sided patches including curved triangular patches was also studied in [9, 10]. In [11] a method to construct a curved triangular patch by combining four interpolation operators: an *interior interpolation operator* and three *side-vertex operators* [5] is presented. The constructed triangular patch reproduces polynomial surfaces of degree four. Another method proposed recently [12] constructs a triangular patch by a *basic approximation operator* and an *interpolation operator*. The constructed triangular patch satisfies C^1 boundary condition and reproduces polynomial surfaces of degree five.

The above methods all work on the assumption that the interpolation conditions on the boundary of the triangle are defined on the same parameter space. In practice, however, this is usually not the case. It is therefore necessary to have a method to determine suitable interpolation conditions so that the methods [1]-[12] can be used directly to construct triangular patches. In [13], a method is presented to construct the cross-boundary conditions. The constructed cross-boundary conditions have suitable magnitudes, but not suitable directions on the boundary of the triangle. This paper overcomes this problem by presenting a simple but efficient method to construct cross-boundary conditions which have both suitable magnitudes and directions. The combination of the new method and the classic functional triangular patch construction methods [1]-[12] can be used to construct a G^1 parametric triangular patch to connect three surface patches. The constructed parametric triangular patch has the same interpolation precision as the classic methods [1]-[12].

2 Problem description

Suppose $\boldsymbol{P}_i(s_i,t_i) = (x_i(s_i,t_i), y_i(s_i,t_i), z_i(s_i,t_i))$, $(0 \leq s_i, t_i \leq 1)$, $i = 1,2,3$, are three given surface patches, defined on different $s_i t_i$-parametric planes. The three patches meet in the way shown in Figure 1. The goal is to construct a triangular patch $\boldsymbol{P}_T(s,t)$ to connect the three patches $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$, with G^1 continuity. $\boldsymbol{P}_T(s,t)$ and $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$, being G^1 continuous means that they have a common boundary and the normal vectors of them on the common boundary have the same direction.

If these three patches are defined on the same parametric *st*-plane, then the methods for constructing functional triangular patches can be used directly to construct a parametric triangular patch to connect these patches with C^1 continuity. In most applications of CAGD, CG and related areas, however, these three patches usually are not defined on the same parameter space. In this case, one needs to define C^1 boundary conditions by the three patches so that the constructed parametric triangular patch can smoothly connect these patches with a "visually pleasing shape" suggested by these three patches. After the C^1 boundary conditions are defined, the functional methods of constructing triangular patches can be used to construct parameter triangular patch directly. As $\boldsymbol{P}_T(s,t)$ and $\boldsymbol{P}_i(s_i,t_i)$, $i = 1,2,3$, are defined on different parameter spaces, $\boldsymbol{P}_T(s,t)$, satisfying C^1 boundary conditions, will connect these three patches with G^1 continuity.

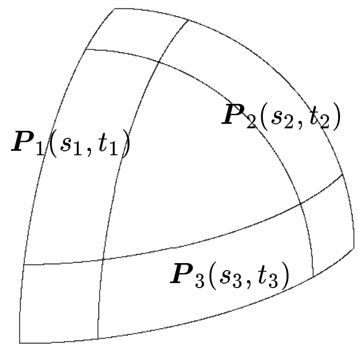


Fig. 1: Three surfaces meet

Let T be an equilateral triangle with vertices $\boldsymbol{v}_1 = (0,0)$, $\boldsymbol{v}_2 = (1,0)$ and $\boldsymbol{v}_3 = (1/2, \sqrt{3}/2)$ in the st-parametric space, e_i denote the opposite side of $\boldsymbol{v}_i$ and τ_i is the unit outward normal vector of e_i, as shown in Figure 2. Let σ_1 denote the unit vector from $\boldsymbol{v}_2$ to $\boldsymbol{v}_3$. σ_2 and σ_3 are defined similarly. The sides e_i, $i = 1,2,3$, can be parameterized as follows:

$$\begin{aligned} \boldsymbol{e}_1(u) &= (1-u)\boldsymbol{v}_2 + u\boldsymbol{v}_3, \\ \boldsymbol{e}_2(u) &= (1-u)\boldsymbol{v}_1 + u\boldsymbol{v}_3, \qquad 0 \le u \le 1 \\ \boldsymbol{e}_3(u) &= (1-u)\boldsymbol{v}_1 + u\boldsymbol{v}_2, \end{aligned} \tag{1}$$

The parametric triangular patch $\boldsymbol{P}_T(s,t)$ to be constructed will be defined on the equilateral triangle T, as shown in Figure 2. On the three sides of T, the boundary curve and cross-boundary slope conditions given by the three surfaces, $\boldsymbol{P}_i(s_i, t_i)$, $i = 1,2,3$ are as follows

$$\boldsymbol{P}_i(\boldsymbol{e}_i(u)), \quad \frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u)), \qquad i = 1,2,3 \tag{2}$$

where $\boldsymbol{e}_i(u)$'s are defined in Eq. (1), $\boldsymbol{P}_i(\boldsymbol{e}_i(u))$ and $\frac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u))$ denote the boundary value and the cross-boundary slope of $\boldsymbol{P}_i(s_i, t_i)$ on the side e_i, respectively.

As the boundary conditions (2) cannot be used directly to construct the triangular patch on T, we will use them to define the new boundary conditions. Let the new boundary conditions be

$$\boldsymbol{P}_T(\boldsymbol{e}_i(u)), \qquad \frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u)), \qquad i = 1,2,3. \tag{3}$$

The new boundary conditions (3) should be defined in a way so that if the three patches $\boldsymbol{P}_i(s_i, t_i)$,$i = 1,2,3$ are defined by the same surface $\boldsymbol{P}(s,t)$, but with different parameter spaces, then $\boldsymbol{P}_T(\boldsymbol{e}_i(u))$, $\frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i = 1,2,3$ on the three sides of T in Figure 2 can be defined by $\boldsymbol{P}(s,t)$, i.e., by

$$\left.\begin{aligned} \boldsymbol{P}_T(\boldsymbol{e}_i(u)) &= \boldsymbol{P}(\boldsymbol{e}_i(u)), \\ \frac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u)) &= \frac{\partial \boldsymbol{P}}{\partial \tau_i}(\boldsymbol{e}_i(u)), \end{aligned}\right. , \quad i = 1,2,3 \tag{4}$$

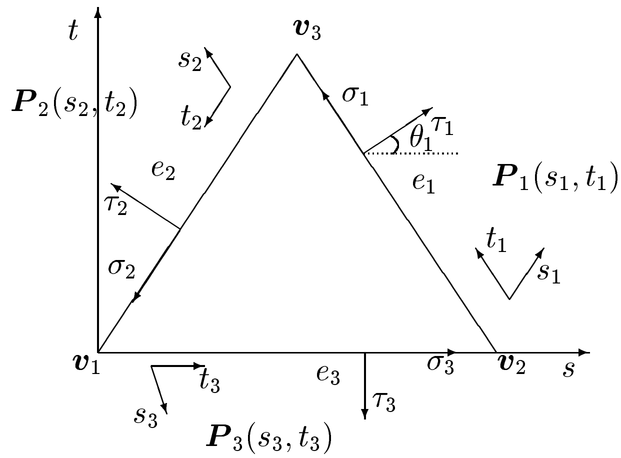


Fig. 2: Three patches meet on T.

3 Constructing the boundary Conditions

We show how to determine $\boldsymbol{P}_T(\boldsymbol{e}_i(u))$, $\dfrac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i = 1, 2, 3$, in this section. As $\boldsymbol{P}_T(s, t)$ and $\boldsymbol{P}_i(s_i, t_i)$ are G^1 continuous on the common boundary, $\boldsymbol{P}_T(\boldsymbol{e}_i(u))$, $\dfrac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u))$, $i = 1, 2, 3$ can be defined by $\boldsymbol{P}_i(s_i, t_i)$, $i = 1, 2, 3$ as follows:

$$\left.\begin{array}{l} \boldsymbol{P}_T(\boldsymbol{e}_i(u)) = \boldsymbol{P}_i(\boldsymbol{e}_i(u)), \\ \dfrac{\partial \boldsymbol{P}_T}{\partial \tau_i}(\boldsymbol{e}_i(u)) = \alpha_i(\boldsymbol{e}_i(u))\dfrac{\partial \boldsymbol{P}_i}{\partial s_i}(\boldsymbol{e}_i(u)) + \beta_i(\boldsymbol{e}_i(u))\dfrac{\partial \boldsymbol{P}_i}{\partial t_i}(\boldsymbol{e}_i(u)) \end{array}\right., \quad i = 1, 2, 3 \tag{5}$$

where $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ are functions of u to be constructed, respectively.

For simplicity, we shall show the construction process of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ only. The $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$, $i = 2, 3$ can be constructed similarly.

As $\dfrac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{e}_1(u))$ and $\dfrac{\partial \boldsymbol{P}_T}{\partial t_1}(\boldsymbol{e}_1(u))$ satisfy

$$\langle \frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{e}_1(u)) \cdot \frac{\partial \boldsymbol{P}_T}{\partial t_1}(\boldsymbol{e}_1(u)) \rangle = 0$$

where $\langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$ denotes the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
It follows from (5) that

$$A_1\alpha_1(\boldsymbol{e}_1(u)) + B_1\beta_1(\boldsymbol{e}_1(u)) = 0 \tag{6}$$

where

$$\begin{array}{l} A_1 = \langle \dfrac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{e}_1(u)) \cdot \dfrac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u)) \rangle, \\ B_1 = \langle \dfrac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u)) \cdot \dfrac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{e}_1(u)) \rangle \end{array}$$

The Eq. (6) gives the function relation between $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$. If one of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ is defined, the rest one is defined. In the following we show how to construct $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$. At point $\boldsymbol{v}_2$, we have

$$\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) = \alpha_1(v_2)\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2) + \beta_1(v_2)\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2). \tag{7}$$

The angle θ_1 between vectors τ_1 and t_3 is 30^o, thus

$$\frac{\partial \boldsymbol{P}_3}{\partial t_3}(\boldsymbol{v}_2) = \frac{\sqrt{3}}{2}\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) - \frac{1}{2}\frac{\partial \boldsymbol{P}_T}{\partial \sigma_1}(\boldsymbol{v}_2).$$

From

$$\frac{\partial \boldsymbol{P}_T}{\partial \sigma_1}(\boldsymbol{v}_2) = \frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2),$$

we have

$$\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2) = \frac{2\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_3}{\partial t_3}(\boldsymbol{v}_2) + \frac{\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2). \tag{8}$$

It follows from Eq. (7) and Eq. (8) that $\alpha_1(v_2)$ and $\beta_1(v_2)$ in Eq. (5), denoted α_1^0 and β_1^0, can be determined by the following equations.

$$\begin{array}{l} \langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\rangle\alpha_1^0 + \langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\rangle\beta_1^0 = \langle\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\rangle, \\ \langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\rangle\alpha_1^0 + \langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_2)\rangle\beta_1^0 = 0 \end{array} \tag{9}$$

On the other hand, at $\boldsymbol{v}_3$ we have

$$\begin{array}{l} \frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_3) = \alpha_1(v_3)\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3) + \beta_1(v_3)\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3), \\ \frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_3) = -\frac{2\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_2}{\partial t_2}(\boldsymbol{v}_3) - \frac{\sqrt{3}}{3}\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3). \end{array} \tag{10}$$

Thus $\alpha_1(v_3)$ and $\beta_1(v_3)$ in Eq. (5), denoted α_1^1 and β_1^1, can also be determined by the following equations.

$$\begin{array}{l} \langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\rangle\alpha_1^1 + \langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\rangle\beta_1^1 = \langle\frac{\partial \boldsymbol{P}_T}{\partial \tau_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\rangle, \\ \langle\frac{\partial \boldsymbol{P}_1}{\partial s_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\rangle\alpha_1^1 + \langle\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\cdot\frac{\partial \boldsymbol{P}_1}{\partial t_1}(\boldsymbol{v}_3)\rangle\beta_1^1 = 0. \end{array} \tag{11}$$

Now $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ can be defined by a linear interpolation as follows:

$$\begin{array}{l} \alpha_1(\boldsymbol{e}_1(u)) = (1-u)\alpha_1^0 + u\alpha_1^1, \\ \beta_1(\boldsymbol{e}_1(u)) = (1-u)\beta_1^0 + u\beta_1^1. \end{array}, \qquad 0 \le u \le 1 \tag{12}$$

where α_1^i and β_1^i, $i = 0, 1$ are defined by (9) and (11).

Based on (6) and (12), there are two ways to define $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$. They are shown below:

$$\begin{array}{l} \alpha_1(\boldsymbol{e}_1(u)) = (1-u)\alpha_1^0 + u\alpha_1^1, \\ \beta_1(\boldsymbol{e}_1(u)) = -A_1\alpha_1(\boldsymbol{e}_1(u))/B_1. \end{array}, \qquad 0 \le u \le 1 \tag{13}$$

$$\begin{array}{l} \beta_1(\boldsymbol{e}_1(u)) = (1-u)\beta_1^0 + u\beta_1^1, \\ \alpha_1(\boldsymbol{e}_1(u)) = -B_1\beta_1(\boldsymbol{e}_1(u))/A_1. \end{array}, \qquad 0 \le u \le 1 \tag{14}$$

The final defination of $\alpha_1(\boldsymbol{e}_1(u))$ and $\beta_1(\boldsymbol{e}_1(u))$ is formed by the combination of (13) and (14), i.e., by

$$
\begin{aligned}
\alpha_1(\boldsymbol{e}_1(u)) &= \frac{(A_1\alpha_1^0 - B_1\beta_1^0)(1-u) + (A_1\alpha_1^1 - B_1\beta_1^1)u}{2A_1} \\
\beta_1(\boldsymbol{e}_1(u)) &= \frac{(B_1\beta_1^0 - A_1\alpha_1^0)(1-u) + (B_1\beta_1^1 - A_1\alpha_1^1)u}{2B_1}
\end{aligned} \tag{15}
$$

Similarly, one can define $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ for $i = 2, 3$ as follows:

$$
\begin{aligned}
\alpha_2(\boldsymbol{e}_2(u)) &= \frac{(A_2\alpha_2^0 - B_2\beta_2^0)(1-u) + (A_2\alpha_2^1 - B_2\beta_2^1)u}{2A_2} \\
\beta_2(\boldsymbol{e}_2(u)) &= \frac{(B_2\beta_2^0 - A_2\alpha_2^0)(1-u) + (B_2\beta_2^1 - A_2\alpha_2^1)u}{2B_2} \\
\alpha_3(\boldsymbol{e}_3(u)) &= \frac{(A_3\alpha_3^0 - B_3\beta_3^0)(1-u) + (A_3\alpha_3^1 - B_3\beta_3^1)u}{2A_3} \\
\beta_3(\boldsymbol{e}_3(u)) &= \frac{(B_3\beta_3^0 - A_3\alpha_3^0)(1-u) + (B_3\beta_3^1 - A_3\alpha_3^1)u}{2B_3}
\end{aligned} \tag{16}
$$

The above construction process of C^1 boundary conditions shows that when the methods for constructing C^1 functional triangular patch are directly applied to the boundary conditions in Eq. (5), a parameter patch $\boldsymbol{P}_T(s,t)$ is constructed, which connects $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$ with G^1 continuity and smooth shape.

4 Discussion

In this section, we will show that the cross-boundary slopes defined by Eqs. (5), (15) and (16) are well defined. To do this, one only needs to prove that if the three surfaces $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$, are defined by the same surface $\boldsymbol{P}(s,t)$ but in different forms, then the new boundary conditions are defined by (4), i.e., by $\boldsymbol{P}(s,t)$. This means that if a method reproduces polynomials of degree n when it is used to construct functional triangular patches, then when it is used with the boundary conditions (5) to construct a parametric triangular patch $\boldsymbol{P}_T(s,t)$, $\boldsymbol{P}_T(s,t)$ will reproduce parametric polynomials of degree n.

Theorem 1 *If surface patches $\boldsymbol{P}_i(s_i,t_i)$, $i = 1, 2, 3$, are defined by the same surface $\boldsymbol{P}(s,t)$, and the transformations from coordinate system st to coordinate system s_it_i are linear, then there exist unique constants c_i and d_i satisfying the following conditions*

$$
\begin{aligned}
\alpha_i &= 1/c_i, \\
\beta_i &= -d_i/c_i
\end{aligned} \tag{17}
$$

where α_i and β_i satisfy $\alpha_i(\boldsymbol{e}_i(u)) = \alpha_i$ and $\beta_i(\boldsymbol{e}_i(u)) = \beta_i$, which means $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ in Eq. (5) are constants in this case.

Proof Only the case $i = 1$ will be considered. The other two cases can be handled similarly. Let $\boldsymbol{V}$ be any point in parametric space, in $\tau_1\sigma_1$ and s_1t_1 coordinate systems, the coordinates of $\boldsymbol{V}$ be (τ_1, σ_1) and (s_1, t_1), respectively. As the transformation from coordinate system st to coordinate system s_1t_1 is linear, so the relationship between (τ_1, σ_1) and (s_1, t_1) can be written as

$$
\begin{aligned}
\tau_1 &= c_1 s_1, \\
\sigma_1 &= d_1 s_1 + t_1.
\end{aligned} \tag{18}
$$

As $\boldsymbol{P}_i(s_1, t_1)$ is defined by $\boldsymbol{P}(s, t)$, it follows from Eq. (18) that $\boldsymbol{P}_1(s_1, t_1)$ can be expressed as

$$\boldsymbol{P}_1(s_1, t_1) = \boldsymbol{P}(c_1 s_1, d_1 s_1 + t_1) = \boldsymbol{P}(\tau_1, \sigma_1).$$

Now

$$\begin{aligned} \frac{\partial \boldsymbol{P}_1(s_1, t_1)}{\partial s_1} &= c_1 \frac{\partial \boldsymbol{P}(\tau_1, \sigma_1)}{\partial \tau_1} + d_1 \frac{\partial \boldsymbol{P}(\tau_1, \sigma_1)}{\partial \sigma_1}, \\ \frac{\partial \boldsymbol{P}_1(s_1, t_1)}{\partial t_1} &= \frac{\partial \boldsymbol{P}(\tau_1, \sigma_1)}{\partial \sigma_1} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial \boldsymbol{P}(\tau_1, \sigma_1)}{\partial \tau_1} &= \frac{1}{c_1} \frac{\partial \boldsymbol{P}(s_1, t_1)}{\partial s_1} - \frac{d_1}{c_1} \frac{\partial \boldsymbol{P}(s_1, t_1)}{\partial t_1}. \\ \frac{\partial \boldsymbol{P}(\tau_1, \sigma_1)}{\partial \sigma_1} &= \frac{\partial \boldsymbol{P}(s_1, t_1)}{\partial t_1} \end{aligned}$$

and this completes the proof of the theorem.

Theorem 1 shows that if surfaces $\boldsymbol{P}_i(s_i, t_i)$, $i = 1, 2, 3$, are defined by the same surface, then α_1^0 and β_1^0 in Eq.(9) and α_1^1 and β_1^1 in Eq. (11) satisfy $\alpha_1^0 = \alpha_1^1$ and $\beta_1^0 = \beta_1^1$, so the functions $\alpha_i(\boldsymbol{e}_i(u))$ and $\beta_i(\boldsymbol{e}_i(u))$ in Eq.(5), $i = 1, 2, 3$, are uniquely determined, i.e., determined by Eq.(40). Consequently, the interpolation conditions are determined uniquely, thus the triangular patch to be constructed is determined uniquely. Therefore the following theorem follows.

Theorem 2 *If the method of constructing functional triangular patch reproduces polynomials of degree n, and the method is directly applied on the interpolation conditions in Eq.(5), then the constructed parametric triangular patch $\boldsymbol{P}_T(s, t)$ reproduces parametric polynomials of degree n.*

5 Experiment

Experiment results presented in this section are carried out by constructing a parametric triangular patch to connect three four-sided patches. The triangular patches are produced by Nielson's method [5]. In Figures 3 and 4, the triangular patch in (a) is produced by directly applying Nielson's method [5] on the boundary curves and cross-boundary slopes defined by the three rectangle patches. The triangular patches in (b) and (c) are produced by using the method presented in [13] and the technique presented in this paper, respectively, to redefine the cross-boundary slopes taken from the three given rectangular patches, then applying Nielson's method [5] on the boundary curves and the redefined cross-boundary slopes. In Figures 3 and 4, some portions of the surfaces on the common boundary of the triangular patch with the three rectangular patches are visually not very smooth. This is the result of Mach band phenomenon. Figures 3 and 4 show that surfaces in (c) have less Mach band phenomenon than those of (b).

Highlight lines [14] have been proved to be effective tool in assessing the quality of a surface. In Figures 5 and 6, the highlight line model is used to compare the above three methods. The figures in Figure 5 are highlight lines of the horizontal fillets of the surfaces in Figure 3. The figures in Figure 6 are highlight lines of the horizontal fillets of the surfaces in Figure 4. The figures in Figures 5 and 6 show that the new method gets better results than the other two methods.

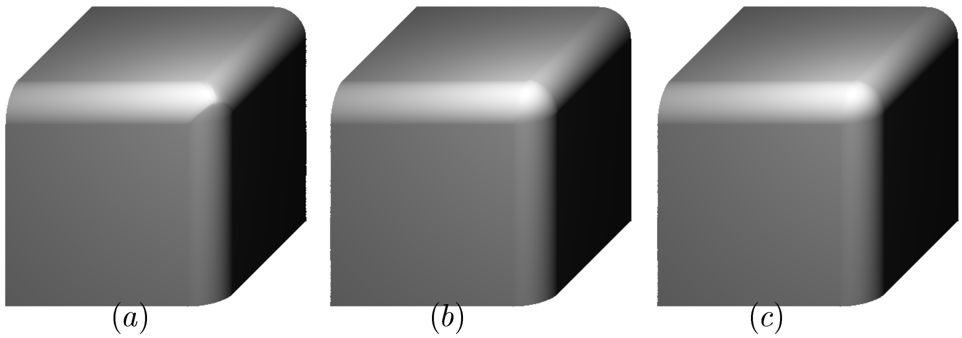


Fig. 3: Example 1

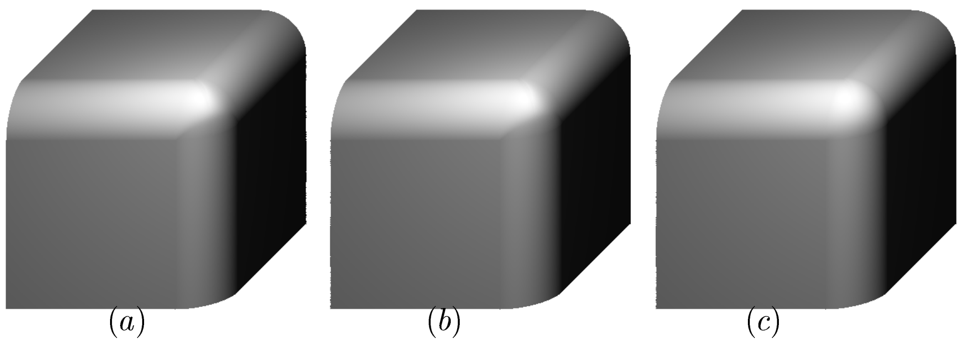


Fig. 4: Example 2

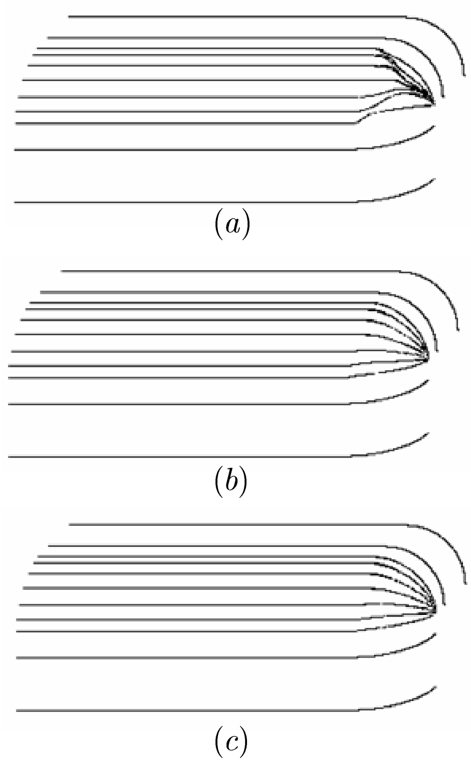
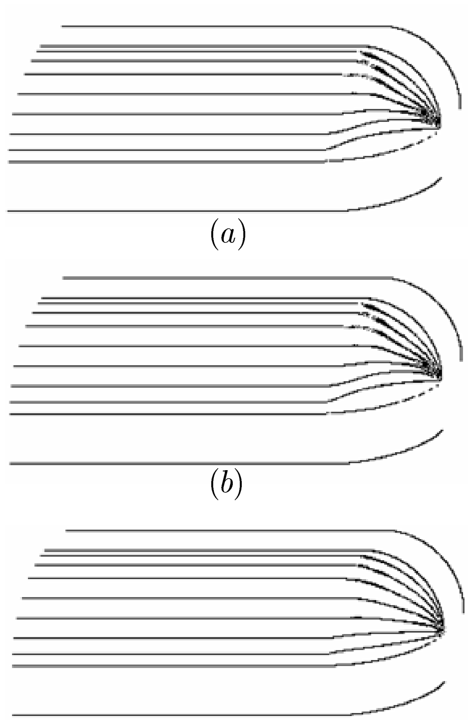


Fig. 5: Example 3



(a)

(b)

(c)

Fig. 6: Example 4

6 Conclusions

A new method that uses functional triangular patch construction methods to construct parametric priangular patches is presented. The new method improves previous methods in both surface shape and surface quality. This is testified by examining Mach band effect and highlight line models of the resulting surface patches. The key in achieving the improvement is a technique to define the cross-boundary conditions. The resulting cross-boundary conditions have not only suitable magnitudes but suitable directions as well.

With the new method, one can directly apply any of the classic functional triangular patch construction methods to construct a C^1 parametric triangular patch to smoothly connect three surface patches. The new method preserves precision of the classic methods. If the applied classic method has a precision of polynomials of degree n, then the constructed parametric triangle patches have a precision of parametric polynomials of degree n.

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