First order absolute moment of Meyer-König and Zeller operators and their approximation for some absolutely continuous functions[1]

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Abstract A sharp estimate is given for the first order absolute moment of Meyer-König and Zeller operators M_n. This estimate is then used to prove convergence of approximation of a class of absolutely continuous functions by the operators M_n. The condition considered here is weaker than the condition considered in a previous paper and the rate of convergence we obtain is asymptotically the best possible.

1 Introduction

For a function f defined on [0, 1], the Meyer-König and Zeller operators M_n [5] are defined by

$$M_n(f,x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \qquad 0 \le x < 1,$$

$$M_n(f,1)=f(1), \qquad m_{n,k}(x)=\binom{n+k}{k} x^k(1-x)^{n+1}. \tag{1}$$

Let

$$K_{n,x}(t)=\begin{cases} \sum_{k\le nt/(1-t)} m_{n,k}(x), & 0<t<1, \\ 1, & t=1, \\ 0, & t=0. \end{cases}$$

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Then operators M_n have the following Lebesgue-Stieltjes integral representation

$$M_n\left(f,x\right)=\int_0^1 f(t)d_tK_{n,x}(t). \tag{2}$$

Estimates of the first order absolute moment of the approximation operators play a key role in various investigations of convergence of the approximation operators (for example, cf. [3, 4, 6, 7, 8]). In this paper we give a sharp estimate for the first order absolute moment of the operators M_n. Furthermore, by means of this estimate and some analysis techniques we establish a convergence theorem on the approximation of a class of absolutely continuous functions by the operators M_n. The rate of convergence we obtain in this theorem is essentially the best possible.

2 Results and Proofs

For the first order absolute moment of Meyer-König and Zeller operators M_n, we have the following result.

Theorem 1. For $x\in(0,1]$, we have

$$M_n(|t-x|,x)=\frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}}+O\left(\frac{1}{n\sqrt{nx}}\right). \tag{3}$$

Proof. If $x=1$, (3) is true. Let $0<x<1$ and write $r=x/(1-x)$. By the fact that $M_n(t,x)=x$ we have

$$\begin{aligned}
M_n(|t-x|,x) &= \sum_{k=0}^{[nr]}\left(x-\frac{k}{n+k}\right)m_{n,k}(x)+\sum_{k=[nr]+1}^{\infty}\left(\frac{k}{n+k}-x\right)m_{n,k}(x)\\
&= 2\sum_{k=0}^{[nr]}\left(x-\frac{k}{n+k}\right)m_{n,k}(x)+M_n(t-x,x)\\
&= 2\sum_{k=0}^{[nr]}\binom{n+k}{k}x^{k+1}(1-x)^{n+1}-2\sum_{k=0}^{[nr]}\frac{k}{n+k}\binom{n+k}{k}x^k(1-x)^{n+1}\\
&= 2\sum_{k=0}^{[nr]}\binom{n+k}{k}x^{k+1}(1-x)^{n+1}-2\sum_{k=0}^{[nr]-1}\binom{n+k}{k}x^{k+1}(1-x)^{n+1}
\end{aligned}$$

$$= \quad 2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1}. \tag{4}$$

Next we estimate

$$2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1}.$$

Using Stirling's formula [9] $n! = \sqrt{2\pi n}(n/e)^n e^{\theta}, \quad 0 < \theta < 1/12n,$ we get

$$2\binom{n+[nr]}{n} = 2\frac{(n+[nr])!}{n!\ [nr]!} = \sqrt{\frac{2}{\pi}}\frac{(n+[nr])^{n+[nr]+1/2}}{n^{n+1/2}\ [nr]^{[nr]+1/2}} e^{\theta_1-\theta_2-\theta_3}, \tag{5}$$

where $0 < \theta_1 < \dfrac{1}{12(n+[nr])}, \ 0 < \theta_2 < \dfrac{1}{12n}, \ 0 < \theta_3 < \dfrac{1}{12[nr]}$.
Set $c(\theta) = \theta_1 - \theta_2 - \theta_3$, simple calculation derives

$$-\frac{1}{12n} - \frac{1}{12[nr]} < c(\theta) \le 0. \tag{6}$$

Since $r = x/(1-x)$, by straightforward calculation we have

$$x^{[nr]+1/2}(1-x)^n = \frac{r^{[nr]+1/2}}{(1+r)^{n+[nr]+1/2}}. \tag{7}$$

Furthermore we find that

$$\frac{(n+[nr])^{n+[nr]+1/2}}{n^{n+1/2}\ [nr]^{[nr]+1/2}} \frac{r^{[nr]+1/2}}{(1+r)^{n+[nr]+1/2}} = \frac{1}{\sqrt{n}}\left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]+1/2}. \tag{8}$$

Thus it follows from (5-8) that

$$\begin{aligned} 2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1} &= \sqrt{x}(1-x)2\binom{n+[nr]}{n} x^{[nr]+1/2}(1-x)^n \\ &= \frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}}\left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]+1/2} e^{c(\theta)}. \end{aligned} \tag{9}$$

Write

$$A(n,r) = \left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]+1/2}, \tag{10}$$

and

$$nr = [nr] + \nu \qquad (0 \le \nu < 1).$$

Then

$$A(n,r) = \left(1+\frac{\nu}{[nr]}\right)^{[nr]+1/2}\left(1+\frac{\nu}{n+[nr]}\right)^{-(n+[nr]+1/2)}.$$

Thus

$$\log A(n,r) = ([nr]+1/2)\log\left(1+\frac{\nu}{[nr]}\right) - (n+[nr]+1/2)\log\left(1+\frac{\nu}{n+[nr]}\right)$$

$$= ([nr]+1/2)\left(\frac{\nu}{[nr]}+O\left(\frac{\nu}{[nr]}\right)^2\right) - (n+[nr]+1/2)\left(\frac{\nu}{n+[nr]}+O\left(\frac{\nu}{n+[nr]}\right)^2\right)$$

$$= O\left([nr]^{-1}\right),$$

which means that

$$A(n,r) = 1 + O\left([nr]^{-1}\right). \tag{11}$$

Hence from (4), (9), (10), (11) and the fact that $e^{c(\theta)} = 1 + O(n^{-1} + [nr]^{-1})$, we get

$$\begin{aligned} M_n(|t-x|,x) &= 2\binom{n+[nr]}{n} x^{[nr]+1}(1-x)^{n+1} \\ &= \frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}}\left(1+O(n^{-1}+[nr]^{-1})\right) \\ &= \frac{\sqrt{2x}(1-x)}{\sqrt{\pi n}} + O\left(\frac{1}{n\sqrt{nx}}\right) \end{aligned}$$

and Theorem 1 is proved.

Next we consider approximation of the operators M_n for a class of absolutely continuous functions Φ_{DB} defined by

$$\Phi_{DB} = \{f \mid f(t)-f(0) = \int_0^t h(u)du,\ t\in[0,1], h \text{ is bounded on } [0,1],$$

$$\text{and } h(x+),\ h(x-) \text{ exist at } x\in(0,1)\}.$$

The following three quantities are needed in this paper. The readers are referred to Reference [8, p. 244] for their basic properties.

$$\Omega_{x-}(h,\delta_1) = \sup_{t\in[x-\delta_1,\ x]} |h(t)-h(x)|, \quad \Omega_{x+}(h,\delta_2) = \sup_{t\in[x,\ x+\delta_2]} |h(t)-h(x)|,$$

$$\Omega(x,h,\lambda) = \sup_{t\in[x-x/\lambda,\ x+(1-x)/\lambda]} |h(t)-h(x)|,$$

where h is bounded on $[0,\,1]$, $x \in [0,1]$ is fixed, $0 \le \delta_1 \le x$, $0 \le \delta_2 \le 1-x$, and $\lambda \ge 1$.

We now state the approximation theorem as follows.

Theorem 2. *Let $f \in \Phi_{DB}$ and write $\mu = h(x+) - h(x-)$. Then for n sufficiently large we have*

$$\left| M_n(f,x) - f(x) - \mu \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \right| \le \frac{4-2x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega(x,h_x,k) + \frac{C|\mu|}{n\sqrt{nx}}. \tag{12}$$

where C is a constant independent of n and x, $[\sqrt{n}]$ is the greatest integer not exceeding $\sqrt{n}$ and $h_x(t)$ is defined by

$$h_x(t) = \begin{cases} h(t) - h(x+), & x < t \le 1 \\ 0, & u = x \\ h(t) - h(x-), & 0 \le t < x, \end{cases} . \tag{13}$$

In view of the fact that $\dfrac{1}{\sqrt{n}} \displaystyle\sum_{k=1}^{[\sqrt{n}]} \Omega(x,h_x,k) \to 0 \ \ (n \to \infty)$, from Theorem 2 we get the asymptotic formula

$$M_n(f,x) = f(x) + \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}}\mu + o(n^{-1/2}),$$

if f satisfies the assumptions of Theorem 2. In particular, (12) is true for $f \in DBV[0,1]$ (that is, f is differentiable function whose derivative is of bounded variation, cf. [3]), since the class of functions $DBV[0,1]$ is a subclass of the class Φ_{DB}. We also point out that Abel [1] presented the complete asymptotic expansion for the operators M_n under much stronger conditions.

Moreover, it is of interest to consider some further results. Let f satisfy the assumptions of Theorem 2 and $\Omega(x,h_x,\lambda) = O(1/\lambda)^\alpha$ for some $\alpha > 0$. Then from Theorem 2 we get

$$M_n(f,x) = f(x) + \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}}\mu + \begin{cases} O(n^{-(\alpha+1)/2}), & \text{if } 0<\alpha<1 \text{ or } 1<\alpha<2 \\ O(\log\sqrt{n}/n), & \text{if } \alpha = 1 \\ O(n^{-3/2}), & \text{if } \alpha \ge 2 \end{cases} .$$

Proof of Theorem 2.

By Bojanic decomposition we have

$$\begin{aligned} h(u) \;=\; & \tfrac{h(x+)+h(x-)}{2} \;+\; \tfrac{h(x+)-h(x-)}{2} sgn(u-x) \;+\; h_x(u) \\ & +\delta_x(u)\left(h(x)-\tfrac{h(x+)+h(x-)}{2}\right), \end{aligned} \tag{14}$$

where $sgn(u)$ is symbolic function, h_x is as defined in (13), and

$$\delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x \end{cases}.$$

Note that $M_n(t,x) = x$, $\int_x^t sgn(u-x)du = |t-x|$, and $\int_x^t \delta_x(u)du = 0$. From (14) it follows by simple computation that

$$f(t) - f(x) = \int_x^t h(u)du = \frac{h(x+)-h(x-)}{2}|t-x| + \int_x^t h_x(u)du.$$

Thus

$$M_n(f,x) - f(x) = \frac{h(x+)-h(x-)}{2} M_n(|t-x|,x) + M_n\left(\int_x^t h_x(u)du,\ x\right). \tag{15}$$

By Lebesgue-Stieltjes integral representation (2) we have

$$\begin{aligned} M_n\left(\int_x^t h_x(u)du,\ x\right) &= \int_0^1 \left(\int_x^t h_x(u)du\right) d_t K_{n,x}(t) \\ &= L(h,n,x) + Q(h,n,x), \end{aligned} \tag{16}$$

where

$$L(h,n,x) = \int_0^x \left(\int_x^t h_x(u)du\right) d_t K_{n,x}(t),$$

$$Q(h,n,x) = \int_x^1 \left(\int_x^t h_x(u)du\right) d_t K_{n,x}(t).$$

Integration by parts and note that $K_{n,x}(0) = 0$, $h_x(x) = 0$ we have

$$|L(h,n,x)| = \left|\int_0^x K_{n,x}(t) h_x(t) dt\right| \leq \int_0^x K_{n,x}(t) \Omega_{x-}(h_x, x-t) dt \tag{17}$$

$$= \int_0^{x-x/\sqrt{n}} K_{n,x}(t)\Omega_{x-}(h_x, x-t)dt + \int_{x-x/\sqrt{n}}^{x} K_{n,x}(t)\Omega_{x-}(h_x, x-t)dt. \qquad (18)$$

By Lemma 2.1 of [2] there holds inequality

$$M_n((t-x)^2, x) \leq \left(1 + \frac{2x}{n-1}\right) \frac{x(1-x)^2}{n+1}.$$

Using this inequality, for $0 \leq t < x$ we deduce that

$$K_{n,x}(t) \leq \sum_{\frac{k}{n+k} \leq t} m_{n,k}(x) \leq \sum_{\frac{k}{n+k} \leq t} \left(\frac{k/(n+k) - x}{x-t}\right)^2 m_{n,k}(x)$$

$$\leq \frac{M_n\left((u-x)^2, x\right)}{(x-t)^2} \leq \frac{1}{(x-t)^2}\left(1 + \frac{2x}{n-1}\right)\frac{x(1-x)^2}{n+1} \leq \frac{2x(1-x)^2}{n(x-t)^2}.$$

Thus by replacement of variable $t = x - x/u$ we have

$$\begin{aligned} \int_0^{x-x/\sqrt{n}} K_{n,x}(t)\Omega_{x-}(h_x, x-t)dt &\leq \frac{2x(1-x)^2}{n} \int_0^{x-x/\sqrt{n}} \frac{\Omega_{x-}(h_x, x-t)}{(x-t)^2} dt \\ &= \frac{2(1-x)^2}{n} \int_1^{\sqrt{n}} \Omega_{x-}(h_x, x/u) du \\ &\leq \frac{2(1-x)^2}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_{x-}(h_x, x/k). \end{aligned} \qquad (19)$$

On the other hand, by inequality $K_{n,x}(t) \leq 1$ and the monotonicity of $\Omega_{x-}(h_x, \lambda)$, it follows that

$$\int_{x-x/\sqrt{n}}^{x} K_{n,x}(t)\Omega_{x-}(h_x, x-t)dt \leq \frac{x}{\sqrt{n}}\Omega_{x-}\left(h_x, x/\sqrt{n}\right) \leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]} \Omega_{x-}(h_x, x/k). \qquad (20)$$

From (19) and (20) and using the basic property $\Omega_{x-}(h_x, \lambda) \leq \Omega(x, h_x, x/\lambda)$ (cf. [8, p. 244]) we get

$$|L(h, n, x)| \leq \frac{2 - 2x + 2x^2}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega(x, h_x, k). \qquad (21)$$

A similar estimate gives

$$|Q(h, n, x)| \leq \frac{2 - 2x^2}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega(x, h_x, k). \qquad (22)$$

Theorem 2 now follows from Eq. (15), (3), (16), (21), and (22).

3 Asymptotic Optimality of the Estimate in Theorem 2

In this section we show that the estimate in Theorem 2 is essentially the best possible.

Take function $f(t) = |t-1/2| \in \Phi_{DB}$ at point $x = 1/2 \in (0,1)$. Then $f(1/2) = 0$, $r = x/(1-x) = 1$, $h(u) = sgn(u-1/2)$, $h_{1/2}(u) \equiv 0$, $h(x+) - h(x-) = 2$, and (12) becomes

$$\left| M_n(|t-1/2|, 1/2) - \frac{1}{2\sqrt{\pi n}} \right| \leq \frac{2\sqrt{2}C}{n^{3/2}}. \tag{23}$$

On the other hand, by Straightforward computation and Stirling's formula [9]

$$n! = (2\pi n)^{1/2}(n/e)^n e^{\theta}, \quad \left(\frac{1}{12n+1} < \theta < \frac{1}{12n}\right),$$

we get

$$\begin{aligned} M_n\left(|t-1/2|, 1/2\right) &= 2\begin{pmatrix} n+n \\ n \end{pmatrix}\left(\frac{1}{2}\right)^{2n+2} = \frac{(2n)!}{n!n!}\left(\frac{1}{2}\right)^{2n+1} \\ &= \frac{\sqrt{2\pi 2n}(2n/e)^{2n}}{\left(\sqrt{2\pi n}(n/e)^n\right)^2}\left(\frac{1}{2}\right)^{2n+1} e^{\theta_1 - 2\theta_2} = \frac{1}{2\sqrt{\pi n}} e^{\theta_1 - 2\theta_2}, \end{aligned} \tag{24}$$

where

$$\frac{1}{24n+1} < \theta_1 < \frac{1}{24n}, \quad \frac{1}{12n+1} < \theta_2 < \frac{1}{12n}.$$

Simple computation gives

$$\frac{1}{9n} < \frac{2}{12n+1} - \frac{1}{24n} < 2\theta_2 - \theta_1 < \frac{1}{6n} - \frac{1}{24n+1} < \frac{1}{6n}. \tag{25}$$

Thus, from (24) and (25) we have

$$\begin{aligned} \left| M_n(|t-1/2|, 1/2) - \frac{1}{2\sqrt{\pi n}} \right| &= \frac{1}{2\sqrt{\pi n}}\left(1 - e^{\theta_1 - 2\theta_2}\right) = \frac{1}{2\sqrt{\pi n}} \frac{e^{2\theta_2 - \theta_1} - 1}{e^{2\theta_2 - \theta_1}} \\ &> \frac{1}{2\sqrt{\pi n}} \frac{2\theta_2 - \theta_1}{e^{2\theta_2 - \theta_1}} > \frac{1}{2\sqrt{\pi n}} \frac{1/9n}{e^{1/2}} = \frac{1}{18\sqrt{\pi e} n^{3/2}}. \end{aligned} \tag{26}$$

Eqs. (23) and (26) mean that for $f(t) = |t-1/2|$, the following inequality holds

$$\begin{aligned} \frac{3}{n}\sum_{k=1}^{[\sqrt{n}]} \Omega\left(\frac{1}{2}, h_{\frac{1}{2}}, k\right) + \frac{1/18\sqrt{\pi e}}{n\sqrt{n}} &\leq \left| M_n\left(f, \frac{1}{2}\right) - f\left(\frac{1}{2}\right) - \frac{1}{2\sqrt{\pi n}} \right| \\ &\leq \frac{3}{n}\sum_{k=1}^{[\sqrt{n}]} \Omega\left(\frac{1}{2}, h_{\frac{1}{2}}, k\right) + \frac{2\sqrt{2}C}{n\sqrt{n}}. \end{aligned} \tag{27}$$

Inequality (27) shows that the estimate (12) in Theorem 2 is asymptotically optimal.

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