General Equations of Aesthetic Curves and Their Self-affinity

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Abstract

The curve is the most basic design element to determine shapes and silhouettes of industrial products and works for shape designers and it is inevitable for them to make it aesthetic and attractive to improve the total quality of the shape design. If we can find an equation of the aesthetic curves, it is expected that the quality of the curve design improves drastically because we can use them as standards to generate, evaluate, and deform the curves. Harada *et al.* insisted that natural aesthetic curves like birds' eggs and butterflies' wings as well as artificial ones like Japanese swords and key lines of automobiles have such a property that their logarithmic curvature histograms (LCHs) can be approximated by straight lines and there is a strong correlation between the slopes of the lines and the impressions of the curves.

In this paper, we define the LCH analytically with the aim of approximating it by a straight line and propose new expressions to represent an aesthetic curve whose LCH is given exactly by a straight line. Furthermore, we derive general equations of aesthetic curves that describe the relationship between their radii of curvature and lengths. Furthermore, we define the self-affinity possessed by the curves satisfying the general equations of aesthetic curves.

Key words: aesthetic curve; general equations of aesthetic curves; self-similarity; self-affinity

1 Introduction

For industrial designers, the curve is one of the most basic design parts that determine shapes and silhouettes of their products and works. It is necessary to make it aesthetically beautiful and attractive to improve the quality of the industrial design. Harada (1997) pointed out that the logarithmic curvature histograms (LCHs) of aesthetically beautiful curves of nature such as birds' eggs and wings of butterflies as well as those of the artifacts such as

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Japanese swords and key lines of automobiles can be approximated by straight lines. Furthermore, the slopes of the approximated lines are strongly related to the impressions of the curves. However, their definition of the LCH was not mathematically defined in the strict sense and that is done procedurally and numerically.

On the other hand, although Kanaya *et al.* (2003) analytically defined the LCH, their definition does not directly give conditions where the LCH can be approximated by a straight line, or in case where the LCH can be approximated, it can not directly determine the slope of the line. Moreover, if the shapes of the curves are obtained by their images, because only discretized data are available, the LCH graph calculated based on their definition is translated in the vertical direction from the LCH graph calculated by Harada's method as explained Section 3.

Hence, in this paper we propose a method to define the LCH analytically with the intension of approximating it by a straight line and formulate the curve whose LCH is strictly given by a straight line with an arbitrary slope. Furthermore, we derive general equations of aesthetic curves from the relationship between the arc length and the radius of curvature of the curve.

2 Quantification of beauty of curves

Here we describe the definition of the LCH given by Harada (1997) and verify the validity of their method to quantify the beauty of the curve. They assumed that the subjects of their method were 1) planar curve and 2) the curve whose curvature varies monotonically. Hence, they did not deal with the curve whose curvature is constant such as the straight line and the circle [1].

2.1 Logarithmic curvature histogram

At first, we make an LCH according to the method proposed by Harada (1997). An image such as shown in Fig. 1(a) is binarized and the points on the sword curve are sampled discretely. These points are approximated by a B-spline curve as shown in Fig. 1(b) and the radius of curvature at an arbitrary position on the curve is estimated.

The total length of the curve is denoted by S_{all} and the radius of curvature at a sampling point a_i is ρ_i. The sampling points $(a_1, a_2, \cdots, a_n)$ are extracted by

[1] Although the line and the circle are beautiful, their beauty originates from their simplicity and we do not deal with these curves.

rhythm	α	elementary functions	impressions
simple	$-$	sin, cubic curve	sharp, strong
	0	(not found)	stable
	$+$	parabola, log, logarithmic spiral	gathering centripetal
complex	$+ \to -$	sin	diverge to converge
	$- \to +$	(not found)	converge to diverge

Table 1
LCH lines' slopes α and their impressions

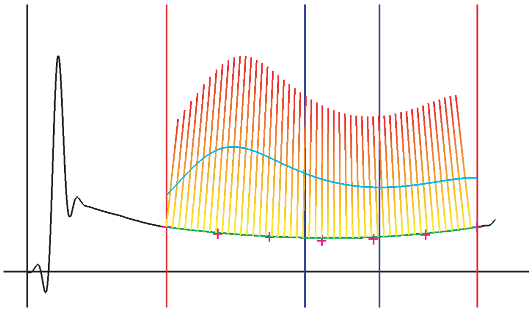
the same interval and the radius of curvature data $(\rho_1, \rho_2, \cdots, \rho_n)$ are obtained by calculating the radius of curvature at each a_i of the approximated B-spline curve.

For example, if the total length of the curve is 100mm and the sampling interval is 0.1mm, the number of the sampling points is 1001. Then the logarithm of the radius of curvature interval $\bar{\rho}_j$ is given by, for instance, subdividing by 100 the interval $[-3, 2]$, the common logarithm of $[0.001, 100]$. The number N_j is calculated by counting ρ_i/S_{all} whose logarithm included in the interval $\bar{\rho}_j$ and the partial arc length s_j(=the interval of the sampling points$\times N_j$) is obtained. Furthermore, the frequency of length $\bar{s}_j$ is determined as the logarithm of the ratio of s_j to $S_{all}(\bar{s}_j = log_{10}(s_j/S_{all}))$. By taking the logarithm of ρ_j in the horizontal direction and $\bar{s}_j$ in the vertical direction, the LCH is drawn as shown in Fig.1(c). In this paper, if the graph of the LCH can be approximated by a straight line, we call it the logarithmic curvature histogram (LCH) line.

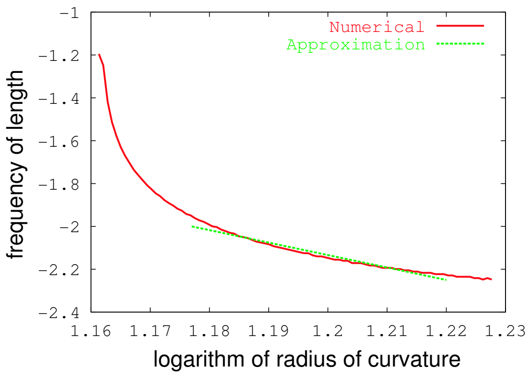
2.2 Impression of the curve

The LCHs of many natural and artificial curves can be approximated by straight lines. Harada (1997) insisted that the impressions of the curves and their slopes of the LCH lines are strongly related and the relationships between the impressions and the slopes can be summarized as described in Table 1. His statements made the characters of the beautiful curves quantitatively clearer than the quantization criteria previously proposed by Higashi *et al.* (1988) that evaluates the monotonic variation of curvature.

(a) Japanese sword



(b) Approximation by a B-spline curve



(c) LCH and its approximation line

Fig. 1. Generation of the LCH

3 Analytically defined logarithmic curvature histogram

As explained in the previous section, Harada's definition of the LCH is not analytical and for example, the frequency of length of the curve can not be evaluated at a certain position on the curve or for a given value of the radius of curvature. In this section, we think about how to define the LCH analytically.

Kanaya *et al.* (2003) showed that for a given curve $\boldsymbol{C}(t) = (x(t), y(t))$, the derivative of the arc length s with respect to the logarithm of the radius of curvature $R = \log \rho$ is given by

$$\frac{ds}{dR} = \frac{(x'y'' - x''y')(x'^2 + y'^2)^{\frac{3}{2}}}{3(x'x'' + y'y'')(x'y'' - x''y') - (x'^2 + y'^2)(x'y''' - x'''y')} \tag{1}$$

where $'$ denotes the derivative with respect to the parameter t. The LCH is mathematically equivalent to the graph whose horizontal and vertical coordinates represent R and $\log|ds/dR|$ respectively.

Equation (1) is enough to analytically define the LCH. However, it does not give any concrete conditions for the parameter ranges where the LCH can be approximated by a straight line or it does not explicitly give the slope of the approximated line. Furthermore, for a curve whose shape is obtained from its image data, only discrete data are available and the partial arc length s_j must be a finite value to calculate the frequency of length. As a result the LCH graph is translated in the horizontal direction from that obtained by Eq. (1) as explained below.

Hence, we think about a transformation of the left side of Eq. (1). Since the LCH graph is expressed by $\log|ds/d(\log\rho)|$ and both of s and ρ are functions of the parameter t,

$$\log\left|\frac{ds}{d(\log\rho)}\right| = \log\left|\frac{\frac{ds}{dt}}{\frac{d(\log\rho)}{dt}}\right| = \log\left|\rho\frac{\frac{ds}{dt}}{\frac{d\rho}{dt}}\right| = \log\rho + \log s_d - \log\left|\frac{d\rho}{dt}\right| \tag{2}$$

where $s_d = ds/dt$. Equation (2) is defined by the radius of curvature and its derivative and describe the relationship between the radius of curvature and the derivative of the arc length more explicitly than Eq. (1).

In the next subsections, we will make analytically clear the relationship between the logarithm of the finite change of the arc length $\Delta s = (\rho\, ds/d\rho)\Delta\log\rho$ and $\log\rho$. We use the parabola and the logarithmic curve as analysis examples to make the discussion more understandable. Then we apply the same analysis to three typical aesthetically beautiful curves: the logarithmic (equiangular) spiral, the clothoid and involute curves. As we will show later, these curves have such a remarkable property that their LCHs can be strictly expressed by straight lines.

3.1 Parabola

We assume that a parabola is given by $\boldsymbol{C}(t = x) = (x, ax^2)$ by letting $t = x$ where a is a positive constant.

If a small change $\Delta\log\rho$ of the logarithm of the radius of curvature ρ is a constant c, then

$$\Delta s = \left|\frac{ds}{d(\log\rho)}\right|\Delta\log\rho = \left|\rho\frac{ds}{d\rho}\right|c \tag{3}$$

By taking the logarithm of both sides of the above equation and by using Eq. (2),

$$\log\Delta s = \log\rho + \log s_d - \log\left|\frac{d\rho}{dx}\right| + \log c \tag{4}$$

For the parabola, ρ and s_d are given by the following expressions:

$$\rho = \frac{(1+4a^2x^2)^{\frac{3}{2}}}{2a}, \quad s_d = (1+4a^2x^2)^{\frac{1}{2}} \tag{5}$$

Therefore, by considering $d\rho/dx = 6axs_d$,

$$\log\Delta s = \log\rho - \log 6ax + \log c. \tag{6}$$

From Eq. (5), x can be expressed by ρ as follows:

$$x = \frac{1}{2a}\{(2a\rho)^{\frac{2}{3}} - 1\}^{\frac{1}{2}}. \tag{7}$$

Since the above expression can be approximated by $x \approx (2a\rho)^{\frac{1}{3}}/2a$ when $(2a\rho)^{2/3} \gg 1$, Equation (6) becomes

$$\log\Delta s = \frac{2}{3}\log\rho + C \tag{8}$$

where $C = -(\log a + \log 2)/3 - \log 3 + \log c$. Hence, the slope of the LCH line is equal to 2/3.

Figure 2 shows the LCH produced by the numerical method mentioned in subsection 2.1 and the line analytically obtained by Eq. (8). From this figure, if x is larger than $x \approx 0.723(\log\rho = 1)$, we can say that Equation (8) approximates the LCH graph very well.

The value of the slope is equal to that mentioned by Harada (1997) and we make clear the condition $(2a\rho)^{2/3} \gg 1$ that is necessary to approximate the graph by a straight line very well. That the LCH graph is given by a straight line means $\Delta s/\rho^\alpha = const$ derived from $\log\Delta s = \alpha\log\rho + C$, or the α-th power of the radius of curvature ρ is proportional to the small change of the arc length Δs.

Based on the above discussion, we find out that the LCH graph defined by Eq. (2) is translated in the vertical direction by $\log c$ from that numerically obtained by using the frequency of length.

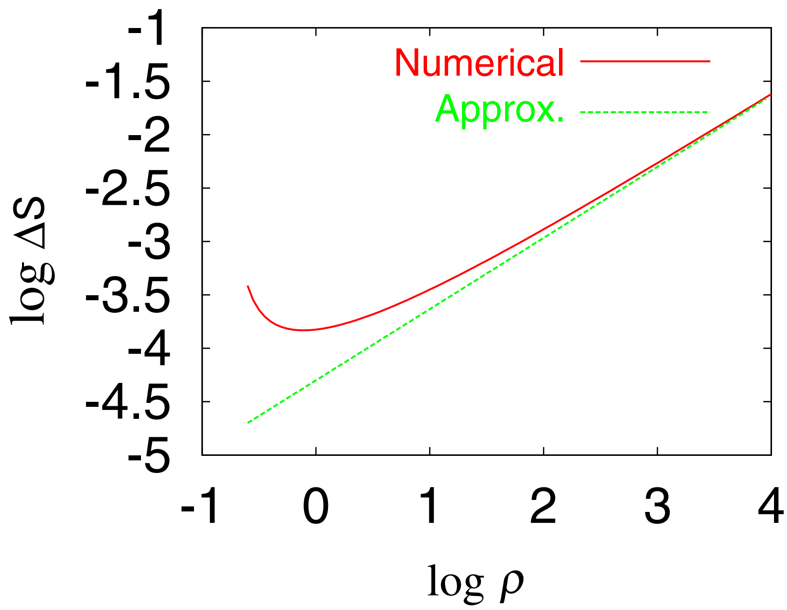


Fig. 2. Numerically obtained LCH of the parabola and its LCH line

3.2 Logarithmic curve

Similar to the parabola, we assume that a logarithmic curve is given by $\boldsymbol{C}(t = x) = (y, \log ax)$ by letting $t = x$ where a is a positive constant. Since $\log ax = \log a + \log x$ and by changing the value of a, the curve moves only along the y axis, we analyze the case where $a = 1$. Here we obtain the LCH directly by using Eq. (2).

The ρ and the s_d of the logarithmic curve are given by

$$\rho = x^2(1 + \frac{1}{x^2})^{\frac{3}{2}}, \quad s_d = (1 + \frac{1}{x^2})^{\frac{1}{2}} \tag{9}$$

By differentiating the ρ in Eq. (9) with respect to x, we obtain

$$\frac{d\rho}{dx} = 2x(1 + \frac{1}{x^2})^{\frac{1}{2}}(1 - \frac{1}{2x^2}) \tag{10}$$

To simplify the above equation, we approximate $1 - 1/2x^2$ by 1. Then

$$\frac{d\rho}{dx} = 2xs_d \tag{11}$$

Furthermore, we assume that $\rho \approx x^2$, i.e. $x \approx \rho^{\frac{1}{2}}$, Equation (2) representing the logarithmic curvature histogram is given by

$$\begin{aligned} \log\left|\frac{ds}{d(\log\rho)}\right| &= \frac{1}{2}\log\rho - \log 2 \\ &= \frac{1}{2}\log\rho + C \end{aligned} \tag{12}$$

where $C = -\log 2$. Hence, the slope of the LCH line is equal to 1/2, which is the same value addressed by Harada (1997). Figure 3 shows the analytically obtained LCH of the logarithmic curve and its LCH line.

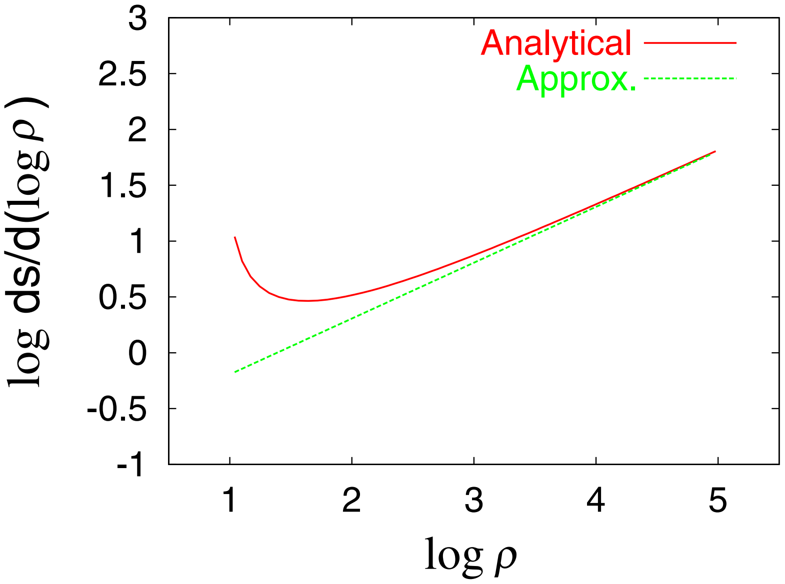


Fig. 3. Analytically obtained LCH of the logarithmic curve and its LCH line

3.3 Logarithmic spiral

We perform the similar analysis for the logarithmic spiral. The logarithmic spiral is also called the equiangular spiral, or Bernoulli's spiral and is well known as a curve representing the shape of the chambered nautilus. It is closely related to the Golden Section that has been regarded as a source of the beauty since the years of the Greeks and the Romans and is one of the typical beautiful curves as discussed in (Livio (2002)). Figure 4 shows an example of the logarithmic spiral.

A logarithmic spiral can be defined in the complex plane by

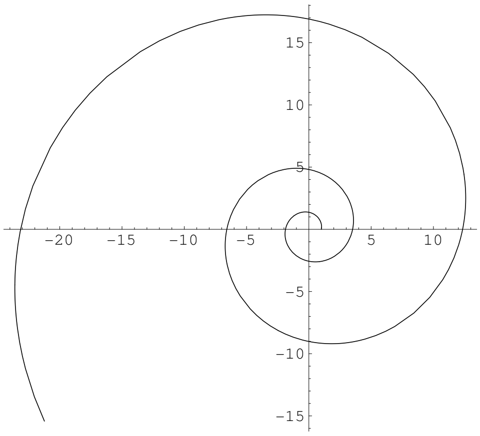


Fig. 4. Logarithmic spiral (a=0.2, b=1)

$$\boldsymbol{C}(t) = e^{(a+ib)t}, \quad (t \geq 0) \tag{13}$$

where i is the imaginary unit and a and b are constants. Its radius of curvature $\rho(t)$ and arc length $s(t)$ are

$$\rho(t) = \frac{1}{b}\sqrt{a^2+b^2}e^{at}, \quad s(t) = \sqrt{a^2+b^2}(e^{at}-1) \tag{14}$$

Hence, Equation (2) for the logarithmic spiral is expressed as follows:

$$\log\left|\frac{ds}{d(\log\rho)}\right| = \log\rho + \log\frac{b}{a}\rho - \log a\rho = \log\rho + C \tag{15}$$

where $C = \log b$. Hence, the LCH of the logarithmic spiral is strictly expressed by a straight line for an arbitrary parameter value. Figure 5 shows the analytically obtained LCH for a given constant $\Delta\log\rho$ and the LCH obtained numerically by the method explained in the subsection 2.1.

3.4 Clothoid curve

The clothoid curve is also called Cornu's spiral and is regarded one of the beautiful curves (for example, see Takanashi (2002)). Figure 6 shows an example of the clothoid curve.

A clothoid curve can be defined in the complex plane by

$$\boldsymbol{C}(t) = \int_0^t e^{iau^2} du \tag{16}$$

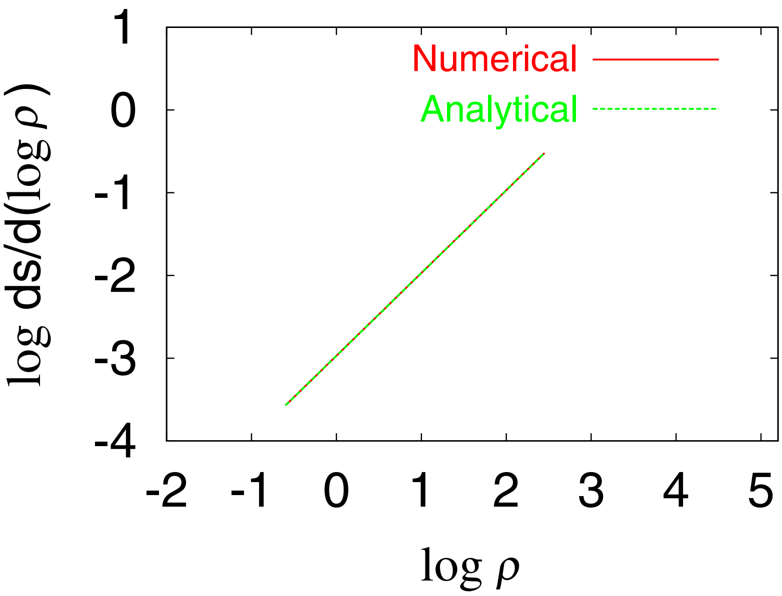


Fig. 5. Numerically and analytically obtained LCHs of a logarithmic spiral

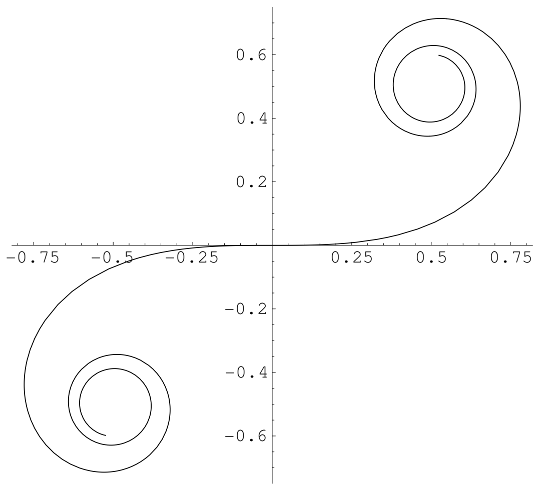


Fig. 6. Clothoid curve (a=1)

where a is a positive constant. The first derivative of $\boldsymbol{C}(t)$ is

$$\frac{d\boldsymbol{C}(t)}{dt} = e^{iat^2} \tag{17}$$

and its absolute value is always equal to 1. Hence, the parameter t is the same as the arc length $s(t)$ (for example, see Farin (2001)). Then the curvature is given by the absolute value of the second derivative

$$\kappa(t) = \left|\frac{d^2\boldsymbol{C}(t)}{dt^2}\right| = |(2iat)e^{iat^2}| = 2at \tag{18}$$

Equation (2) for the clothoid curve is expressed as follows:

$$\log\left|\frac{ds}{d(\log\rho)}\right| = \log\rho - \log\frac{1}{2at^2} = -\log\rho + C \tag{19}$$

where $C = -\log a - \log 2$. Hence, the LCH of the clothoid curve is strictly expressed by a straight line for an arbitrary parameter value. Figure 7 shows the analytically and numerically obtained LCH.

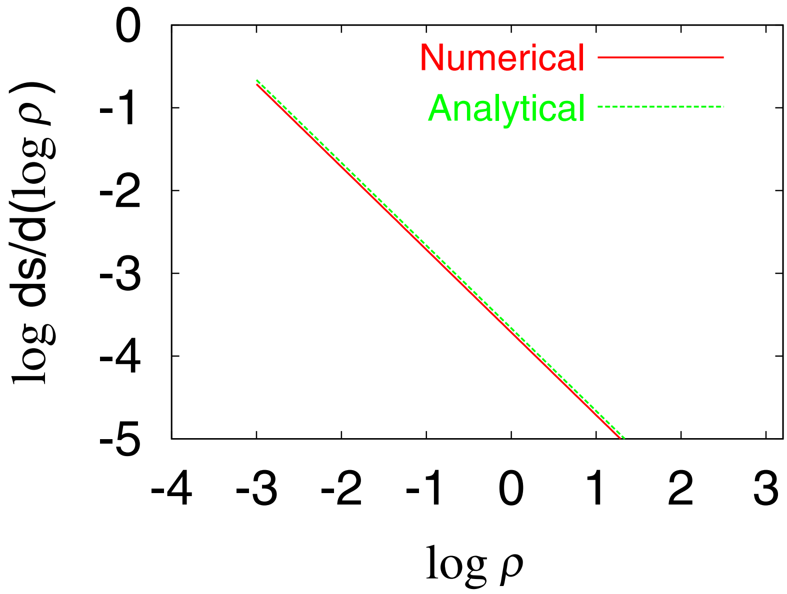


Fig. 7. Numerically and analytically obtained LCHs of a clothoid curve

3.5 Involute curve

The involute curve is used for the design of gear profiles. Figure 8 shows an example of the involute curve.

An involute curve can be expressed using an angle θ as a parameter as $\boldsymbol{C}(t) = (\cos t + t\sin t, \sin t - t\cos t)$ if we omit a global scaling factor. Its radius of curvature $\rho(t)$ and arc length $s(t)$ are given by

$$\rho(t) = \frac{1}{t}, \quad s(t) = t^2 \tag{20}$$

Equation (2) for the involute curve is expressed as follows:

$$\log\left|\frac{ds}{d(\log\rho)}\right| = 2\log\rho \tag{21}$$

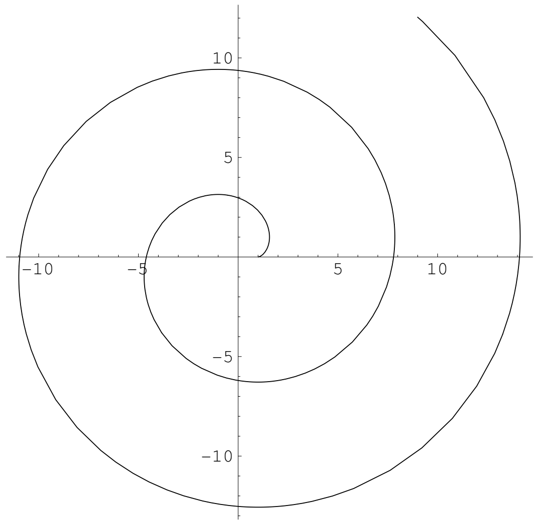


Fig. 8. Involute curve

Hence, the LCH of the involute curve is strictly expressed by a straight line for an arbitrary parameter value. Figure 9 shows the analytically and numerically obtained LCHs.

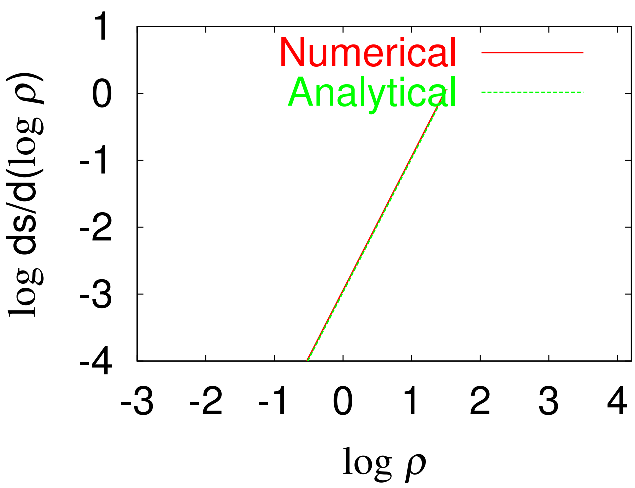


Fig. 9. Numerically and analytically obtained LCHs of an involute curve

3.6 Counterexample

We discuss the properties of the Archimedean spiral whose LCH can not be approximated by a straight line properly as a counterexample of aesthetic curves.

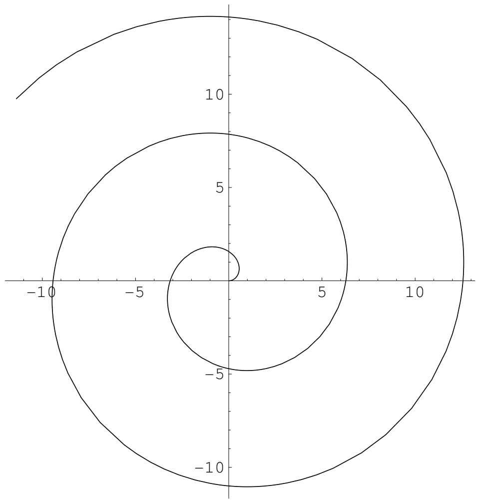


Fig. 10. Archimedean spiral (a=1,b=1)

The Archimedean spiral is also called the uniform spiral and is a spiral whose radius increases in proportion to the angle to the x-axis as shown in Fig.10. In the complex plane, its general expression is given by

$$\boldsymbol{C}(t) = at\, e^{ibt}, \quad (t \geq 0) \tag{22}$$

where a and b are constants.

The LCH of a Archimedean spiral drawn by Eq. (2) is shown in Fig.11. Around the start point, the change of the curvature is slow and the value in the vertical direction is relatively large, but the value rapidly decreases and then rapidly increases after a certain value (around $\log\rho = -0.4$). If the parameter t in Eq. (22) becomes large enough, the curve approximates a circular arc and the change of the radius of curvature becomes slow. Therefore the value in the vertical direction becomes large again.

Although the Archimedean spiral closely approximates an involute curve for large parameter values [2], around its characteristic shape ($-0.6 < \log\rho < 1.5$) the slope of its LCH rapidly changes and it is impossible to approximate it by a straight line properly.

The definition of the Archimedean spiral is simply given by Eq. (22) and has a geometrically regular property that the intersection intervals on the x and y axes are constant. However, the main usage of the Archimedean spiral is for the design of machines such as water pumps and it is not so frequently used for aesthetic design purposes.

[2] Hence, if the parameter t of the Archimedean spiral becomes larger and larger, its slope of the LCH converges to 2. In such a sense, the Archimedean spiral is similar to the parabora and the logarithm curve and it might not be a good counterexample.

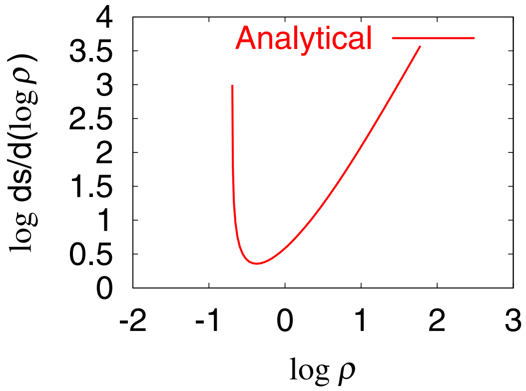


Fig. 11. LCH of an Archimedean spiral (a=1,b=1)

3.7 Curve with an arbitrary LCH line slope

For the design of curves, it is desirable to represent a curve whose LCH line can have an arbitrarily valued slope. Since the radius of curvature of the clothoid curve can be formulated by a simple expression, we consider extensions of the clothoid curve to make them have an arbitrary slope for their LCH lines.

Here we apply the fine tuning method developed by Miura *et al.* (2001) to the clothoid curve and extend its representation. The fine tuning method can scale curvature at a point on curves and surfaces to an arbitrary value. In the curve case, for a given curve $\boldsymbol{C}(t)$, by using a scalar function $g(t) > 0$ and define a new curve as follows:

$$\boldsymbol{C}'(t) = \boldsymbol{P}_0 + \int_0^t g(u)\frac{d\boldsymbol{C}(u)}{du}du \tag{23}$$

Namely differentiate the original curve, scale the first derivative by multiplying a scale function and change the value of curvature arbitrarily. The clothoid curve applied by the fine tuning (Fine Tuned Clothoid : FTC) is defined by the following expression in the complex plane:

$$\boldsymbol{C}(t) = \int_0^t g(u)e^{iau^2}du \tag{24}$$

where a is a constant and $g(t)$ is a scale function whose value is always positive.

By using the radius of curvature ρ_c of the clothoid curve, we define $g(t) = (1/2at)^\beta$ If we assume β can be positive or negative values, $g(t)$ is equivalent to be the $-\beta$-th power of t except for the constant coefficient. The analysis results yield

$$\log \Delta s = \frac{\beta - 1}{\beta + 1} \log \rho + C \tag{25}$$

where $C = -\log(\beta + 1) - \log 2 - \log a + \log c$. Hence, the LCH graph is given by a straight line whose slope is $(\beta - 1)/(\beta + 1)$ and the slope α can be an arbitrary value except for 1 [3]. Figure 12 shows several FTC curves whose LCH lines' slopes are given by α. NOte that the curve whose α is equal to -1 is a clothoid curve.

The FTC curve which has 1 for its LCH line slope can be obtained with $g(t) = c_0 t e^{c_1 t^2}$ by solving a differential equation $\Delta s/\rho = const$ where c_0 and c_1 are constants. In this case, we can perform the integration explicitly and it turns out to be a logarithmic spiral expressed by

$$\boldsymbol{C}(t) = e^{ic_2} \int_0^t c_0 u e^{c_1 u^2} e^{iau^2} du = \frac{c_0}{2(c_1 + ia)} e^{ic_2} e^{(c_1 + ia)t^2} \tag{26}$$

where c_2 is a integration constant.

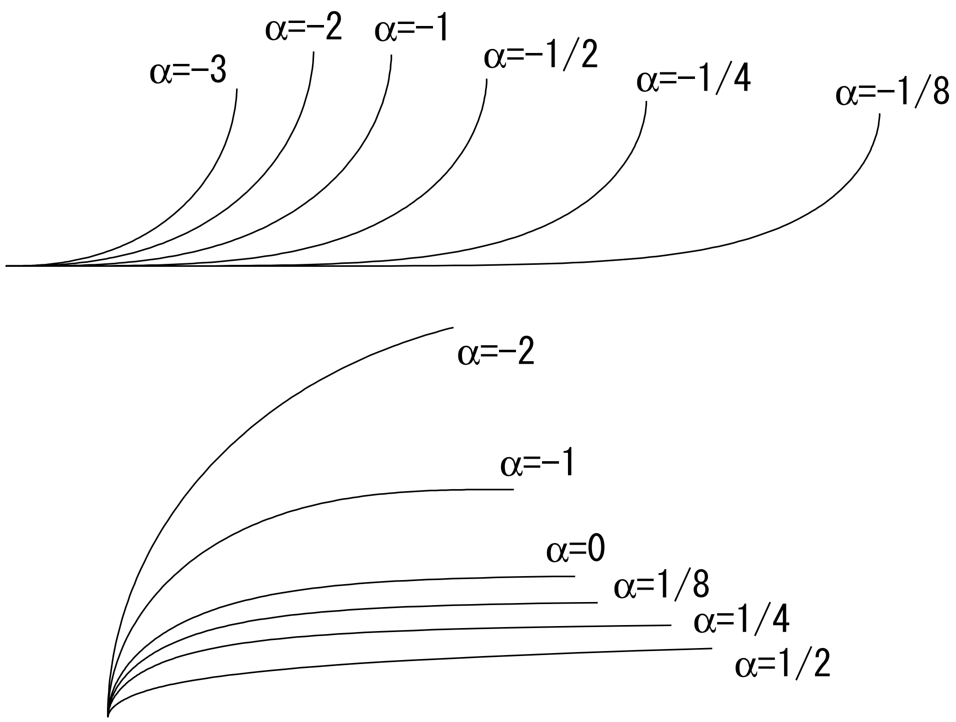


Fig. 12. Curves whose LCH graphs are given by α-sloped straight lines

[3] If β is equal to -1, the curve becomes a circle

4 General equations of aesthetic curves

In this section, we derive equations of the curve whose LCH is strictly given by a straight line. The curve obtained here can represent aesthetic curves and we call it general equations of aesthetic curves.

4.1 Derivation of general equations

If we assume that the LCH graph of a given curve is strictly expressed by a straight line, on the left side of Eq. (2) there is a constant α and

$$\log\left|\rho\frac{ds}{d\rho}\right| = \alpha\log\rho + C \tag{27}$$

where C is a constant. By transforming Eq. (27), we obtain

$$\frac{1}{\rho^{\alpha-1}}\frac{ds}{d\rho} = e^{C} = C_0 \tag{28}$$

Hence,

$$\frac{ds}{d\rho} = C_0\rho^{\alpha-1} \tag{29}$$

If $\alpha \neq 0$,

$$s = \frac{C_0}{\alpha}\rho^{\alpha} + C_1 \tag{30}$$

In the above equation, C_1 is an integral constant. Therefore

$$\rho^{\alpha} = C_2 s + C_3 \tag{31}$$

where $C_2 = \alpha/C_0$ and $C_3 = -(C_1\alpha)/C_0$. Here we rename C_2 and C_3 to c_0 and c_1, respectively. Then

$$\rho^{\alpha} = c_0 s + c_1 \tag{32}$$

The above equation indicates that the α-th power of the radius of curvature ρ is given by a linear function of the arc length s [4]. We call the above equation the first general equation of aesthetic curves in this paper [5].

In case of $\alpha = 0$,

$$s = C_0 \log \rho + C_1 \tag{33}$$

Hence,

$$\rho = C_2 e^{C_3 s} \tag{34}$$

where $C_2 = e^{-C_1/C_0}$ and $C_3 = 1/C_0$. We rename C_2 and C_3 as c_0 and c_1, respectively. We get

$$\rho = c_0 e^{c_1 s} \tag{35}$$

The ρ is given by an exponential function of s. We call the above equation the second general equation of aesthetic curves in this paper.

The logarithmic spiral and clothoid curve are regarded as two typical beautiful curves. One of the principal characters of the logarithmic spiral that its radius of curvature and arc length are proportional is well known and it means that the logarithmic spiral satisfies Eq. (32) and its α is equal to 1. On the other hand the main property of the clothoid curve is that its radius of curvature is in inverse proportion to its arc length. Equation (32) is satisfied for the clothoid curve if α is given by -1.

In summary, the general equation of aesthetic curves expressed by Eq. (32) includes the most typical beautiful curves such as the logarithmic spiral and the clothoid curve.

4.2 Parametric expression of the general aesthetic curves

In this subsection, we find a parametric expression of the general equation of aesthetic curves given by Eq. (32).

[4] Note that the local property that the α-th power of the radius of curvature ρ is proportional to the small change of the arc length Δs is satisfied globally for the whole curve.
[5] If we do not care about how to derive the general aesthetic equation, note that when $c_0 = 0$, it can represent lines and circles whose radii of curvature are constant.

We assume that a curve $\boldsymbol{C}(s)$ satisfies Eq. (32). Then

$$\rho(s) = (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{36}$$

As s is the arc length, $|s_d| = 1$ (refer to, for example, Farin (2001)) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{37}$$

Since $\rho(s) = 1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{-\frac{1}{\alpha}} \tag{38}$$

Hence, if $\alpha \neq 1$,

$$\theta = \frac{\alpha(c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0} + c_2 \tag{39}$$

If the start point of the curve is given by $\boldsymbol{P}_0 = \boldsymbol{C}(0)$,

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{i\frac{\alpha(c_0 u + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0}} du \tag{40}$$

The above expression can be regarded as an extension of the clothoid curve whose power of e in its definition is changed from 2 to $\alpha+1$ and its LCH line's slope can be specified to be equal to any value except for 0.

In Eq. (38) if $\alpha = 1$,

$$\theta = \frac{1}{c_0}\log(c_0 s + c_1) + c_2 \tag{41}$$

Then

$$\begin{aligned} \boldsymbol{C}(s) &= \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{i\frac{\log(c_0 u + c_1)}{c_0}} du \\ &= \boldsymbol{P}_0 + \frac{c_0 - i}{c_0(c_0^2+1)} e^{ic_2} \{ e^{(c_0+i)\frac{\log(c_0 s + c_1)}{c_0}} - e^{(c_0+i)\frac{\log c_1}{c_0}} \} \end{aligned} \tag{42}$$

The above equation expresses a logarithmic spiral.

For the second general equation of aesthetic curves expressed by Eq. (35),

$$\frac{d\theta}{ds} = \frac{1}{c_0}e^{-c_1 s} \tag{43}$$

$$\theta = -\frac{1}{c_0 c_1}e^{-c_1 s} + c_2 \tag{44}$$

Therefore the curve is given by

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2}\int_0^s e^{-\frac{i}{c_0 c_1}e^{-c_1 s}} ds \tag{45}$$

5 Self-Affinity

Harada *et al.* (1995) addressed that the curve whose logarithmic curvature histogram was expressed by a straight line had a self-affinity, but his proof was not mathematically strict. His statement that "the property is called the self-affinity of the curve that the curve obtained by cutting the original curve at two positions and applying such an affine matrix that scales by two different scaling factors in the two orthogonal directions becomes identical to the original curve" is misleading. It might be interpreted that there is a 2×2 matrix depending only on the cutting positions. However, it is trivial that there is not such a matrix for a clothoid curve [6] . It means we need a new definition of the self-affinity for aesthetic curves possessed by those who satisfy the general equations of aesthetic curves.

In the following subsections, we will discuss about the self-similarity: a special case of the self-affinity and show the logarithmic spiral has the self-similarity. Then we will prove that the curves that satisfy one of the general equations of aesthetic curves possess the self-affinity defined in this paper.

5.1 Self-similarity of the logarithmic spiral

The self-similarity is a characteristic property of the fractal geometry and it becomes a similar shape to the original after scaling it like a saw-toothed coastline (Mandelbrot (1983)). We will show that the logarithmic spiral has the self-similarity below.

[6] If we apply an affine matrix to a multi-looped clothoid curve, the curve will warp and not be another clothoid curve.

Similar to subsection 3.3, a logarithmic spiral can be defined in the complex plane by

$$\boldsymbol{C}(t) = e^{(a+ib)t} \quad (t \geq 0) \tag{46}$$

where a and b are constants. We cut the head portion of the curve and define a new curve $\boldsymbol{D}(t)$ for $t \geq 1$ of $\boldsymbol{C}(t)$ as follows:

$$\boldsymbol{D}(t) = \boldsymbol{C}(t+1) = e^{a}e^{ib}\boldsymbol{C}(t) \tag{47}$$

As we see from the above equation, the curve $\boldsymbol{D}(t)$ can be obtained by scaling $\boldsymbol{C}(t)$ by the factor e^a and rotating it by the angle b about the origin. Therefore since the original curve is recovered by scaling the curve whose head portion is cut out, the logarithmic spiral has the self-similarity. Here we removed the head portion where $t < 1$, but it is rather obvious to be able to argue similarly by cutting an arbitrary head portion.

To make clear the relationship between the self-similarity and self-affinity discussed in the next subsection, we show the relationship between the radius of curvature $\rho(t)$ and the arc length $s(t)$ of the spiral curve expressed by Eq. (46). They are given by

$$\rho(t) = \frac{1}{b}\sqrt{a^2+b^2}\, e^{at}, \quad s(t) = \sqrt{a^2+b^2}\,(e^{at}-1) \tag{48}$$

Hence,

$$\rho(t) = c_0 s(t) + c_1 \tag{49}$$

where $c_0 = 1/b$ and $c_1 = -1/(b\sqrt{a^2+b^2})$. That means the logarithmic spiral satisfies the first general equation of aesthetic curves. The radius of curvature $\rho_D(t)$ and the arc length $s_D(t)$ of the curve $\boldsymbol{D}(t)$ are

$$\rho_D(t) = \frac{e^a}{b}\sqrt{a^2+b^2}\, e^{at}, \quad s_D(t) = e^a\sqrt{a^2+b^2}\,(e^{bt}-1) \tag{50}$$

and both of the radius of curvature and the arc length are scaled by e^a.

5.2 Self-affinity of aesthetic curves

Although the self-similarity can be found ubiquitously in the natural world, not so many phenomena are known. Some kind of the Brownian motion has such a self-affinity that by doubling the scale of the time and scaling its amplitude by $\sqrt{2}$, it shows the self-similarity (Takagi (1992)). That means the self-similarity by scaling in the different coordinate axes by different values is called the self-affinity. We will discuss the self-affinity possessed by the curves that satisfy the first and second general equations of aesthetic curves below.

Assume that a curve satisfies the first general equation of aesthetic curves expressed by Eq. (32). Then for a given α,

$$\rho(t)^{\alpha} = c_0 s(t) + c_1 \tag{51}$$

As even if $s(t)$ is reparameterized by an arbitrary monotonously increasing function, the shape remains the same, we reparametrize the curve by $s(t) = c_1(e^{\beta t} - 1)/c_0$. Then

$$\rho(t) = c_1^{\frac{1}{\alpha}} e^{\frac{\beta}{\alpha} t} \tag{52}$$

Similar to the previous subsection, we get a curve $\boldsymbol{D}(t)$ by cutting the head portion of the curve by substituting t with $t+1$ and obtain its radius of curvature $\rho_D(t)$

$$\rho_D(t) = \rho(t+1) = c_1^{\frac{1}{\alpha}} e^{\frac{\beta}{\alpha}} e^{\frac{\beta}{\alpha} t} \tag{53}$$

Hence, the radius of curvature of the curve without the head portion is given by scaling that of the original curve by $e^{\frac{\beta}{\alpha}}$.

The arc length of the curve $s_D(t)$ is

$$s_D(t) = s(t+1) - s(t) = \frac{c_1}{c_0} e^{\beta} (e^{\beta t} - 1) \tag{54}$$

Therefore the arc length of $\boldsymbol{D}(t)$ is obtained by scaling that of the original by e^{β}.

In summery, the curve without the head portion is identical with that generated by scaling the radius of curvature of the original curve in the principal normal direction by $e^{\beta/\alpha}$ and its arc length in the tangent direction by e^{β}, or in the Frenet frame. This means that at an arbitrary point on the curve in the two different orthogonal directions, the principal normal and the tangent by scaling the cut curve by the different factors, the original curve can be obtained. We define this kind of the self-affinity as that of aesthetic curves.

It is easy to show that a curve satisfying the second general equation of aesthetic curves has the self-affinity of aesthetic curves as follows. Assume that

$$\rho(t) = c_0\, e^{c_1 s(t)} \tag{55}$$

We reparameterize by $s(t) = t$ and shift the parameter by 1 from t to $t + 1$

$$\rho(t+1) = c_0\, e^{c_1(t+1)} = c_0\, e^{c_1}\, e^{c_1 t} \tag{56}$$

Hence, the radius of curvature of the curve without the head portion is obtained by scaling that of the original curve by e^{c_1}. Although the arc length remains the same, the curve is scaled in the principal normal direction and the curve has the same self-affinity as that satisfying the first general equation.

The self-affinity of the Brownian motion introduced in this subsection is for a fixed coordinate system made by the time and the amplitude axes, but the self-affinity of aesthetic curves is for the Frenet frame along the curve. Although the matrix used for the affine transformation is the same in the moving coordinate system, no affine matrix exists for a fixed coordinate system.

6 Conclusion

Harada's work is very suggestive to analyze the characteristics of aesthetic curves. In this paper, based on his work we have defined the LCH analytically with the purpose of approximating it by a straight line and formulated the curve whose LCH graph is strictly expressed by a straight line. Furthermore, we have found out two kinds of the relationship between the radius of curvature and the arc length of the curve whose LCH graph is given by a straight line and proposed them as general equations of aesthetic curves. We have shown that the curve satisfying either the first or second general equations of aesthetic curves has some kind of a self-affinity and defined it as the self-affinity of aesthetic curves.

For future work, we are planning an automatic classification of curves: 1) determine the rhythm to be simple (monotonic) or complex (consisting of plural rhythms), 2) calculate the slope of the line approximating the LCH graph. We think there are a lot of possibilities to use the general aesthetic equations to many applications in the fields of computer aided geometric design. For example, we may be able to apply the equations to deform curves to change their impressions, say, from sharp to stable. Another example is smoothing for reverse engineering. Even if only noisy data of curves are available, we may be able to use the equations as kinds of rulers to smooth out the data and yield aesthetically high quality curves. We will develop a CAD system using the equations.

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