CS375: Logic and Theory of Computing

Fuhua (Frank) Cheng

Department of Computer Science

University of Kentucky

Question 1:

Q: Why are you Here? or,

why are you taking this course?

Ans: ???

Question 2:

Q: Who do you think is the most important scientist from the last (20-th) century?

Ans: keep your answer to yourself.

The same question will be asked at the end of the semester again.

For the first question: why are you here?

You are here to learn three things.

Three things you will learn in this class:

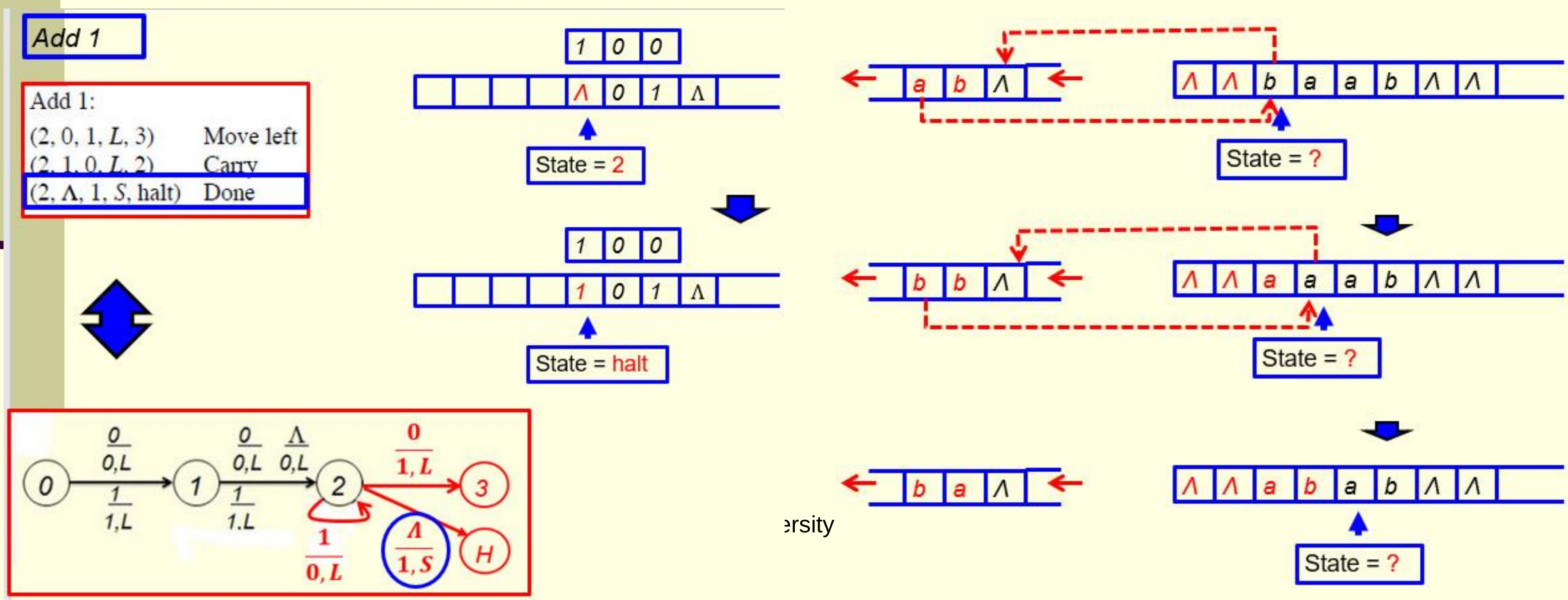
1: The foundation of modern day computers

i.e., Turing machines (1936)

- *a piece of work that changed the world/ human history*

1: The foundation of modern day computers

Such as: *how to design a machine that can perform arithmetic operations?*
or, *how to design a machine that can process/parse strings?*



Three things you will learn in this class:

2: The person who developed that theory

Dr. Alan Turing

- *The Nobel Prize equivalent in computer science area (Turing Award) was named after him*

Three things you will learn in this class:

3: A little history on the life of that person

Dr. Alan Turing (1912-1954)

- *including the reason why the logo of the Apple Computer company is a bitten apple*

The Arrangement:

Preliminaries

↓

Regular Languages + Finite Automata

↓

Context-Free Languages + Pushdown Automata

↓

Turing machines + Church-Turing Thesis

Table of Contents:

- **Week 1: Preliminaries (set algebra, relations, functions) (read Chapters 1-4)**
- **Weeks 2-5: Regular Languages, Finite Automata (Chapter 11)**
- **Weeks 6-8: Context-Free Languages, Pushdown Automata (Chapters 12)**
- **Weeks 9-11: Turing Machines (Chapter 13)**

Table of Contents (conti):

- **Weeks 12-13: Propositional Logic (Chapter 6), Predicate Logic (Chapter 7), Computational Logic (Chapter 9), Algebraic Structures (Chapter 10)**

1. Preliminaries – set algebra

- **Set** : collection of things

  (order not important; repetition not allowed)

- Notations: $x \in S$, $x \notin S$

  $S = \{x_1, x_2, x_3, \ldots, x_n\}$

  { }, Φ : empty set

  Z, N, Q, R

- Notations:

A = B : *two sets A and B are equal*

$\{a,b,c\}=\{c,b,a\}$?

$\{a,a,b,c\}=\{a,b,c\}$?

$\{x \mid P\}$: *the set of all x that satisfies P*

e.g., the set of odd natural numbers = {1,3,5, … }

$= \{x \mid x = 2k+1 \;\; for \;\; some \;\; k \in N\}$

- Notations:

$A \subseteq B$: *A is a subset of B*

$N \subseteq Z \subseteq Q \subseteq R$

$S \subseteq S$

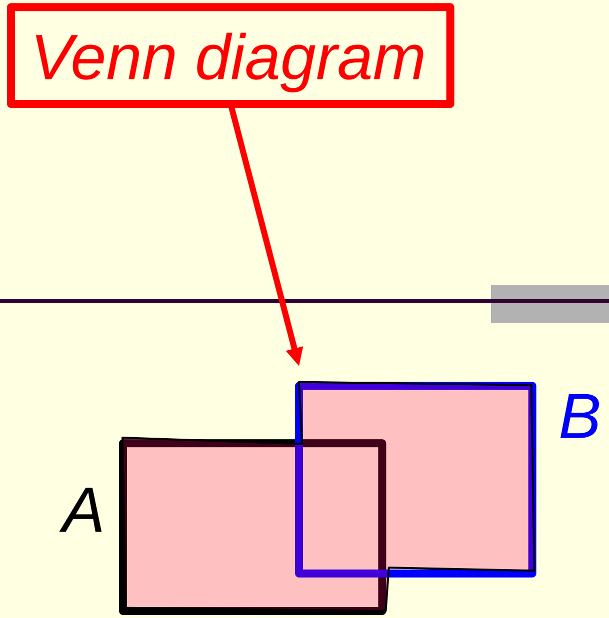
$\Phi \subseteq S$

Power(S) = the set of all subsets of S

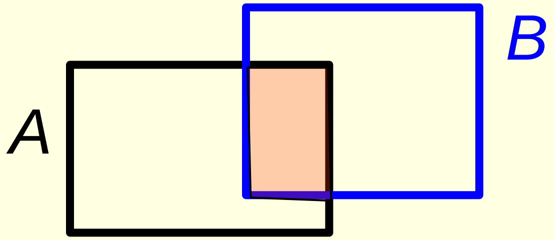
power({a, b, c}) = ?

If $|S| = n$ *then* $|power(S)| = 2^n$

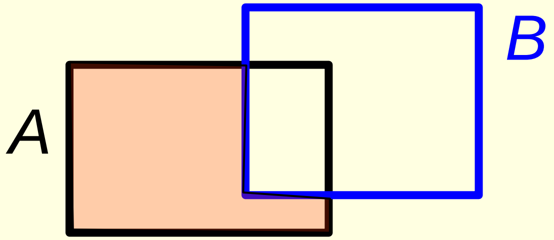
- Notations:



$A \cup B$: union



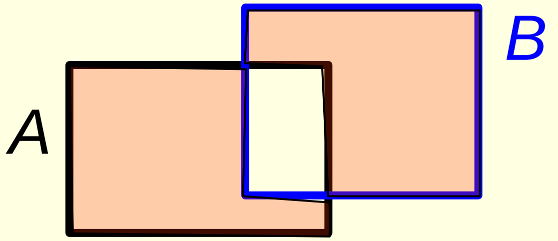
$A \cap B$: intersection



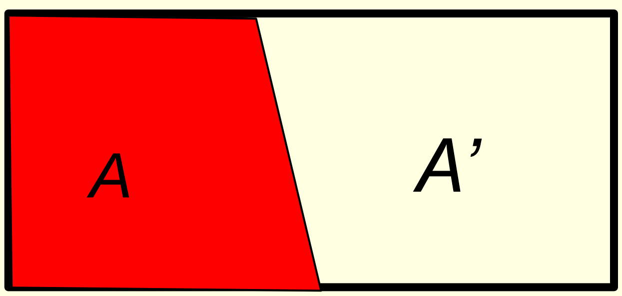
$A - B$: difference

Notations:

$A \oplus B$: *symmetric difference*



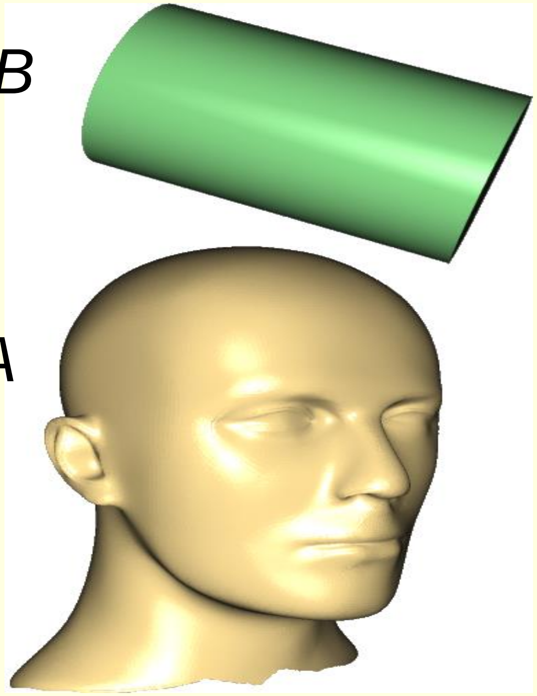
$A' = U - A$: *universal complement*



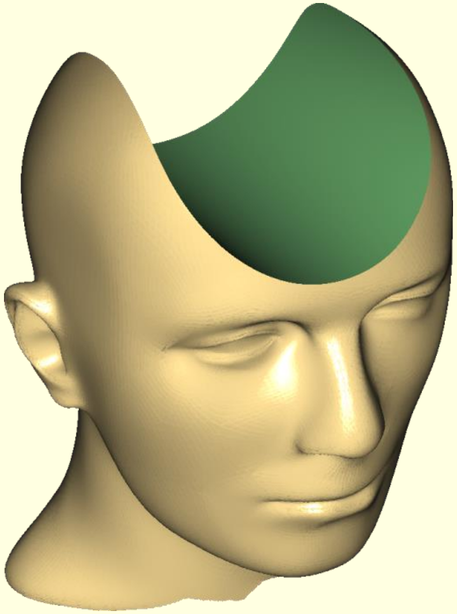
e.g.,

B

A



A - B
(depends on where B is)



Properties:

Union *and* ***intersection*** *are* ***commutative***, ***associative***, *and* ***distribute*** *over each other*

Absorption:

$$A \cup (A \cap B) = A$$

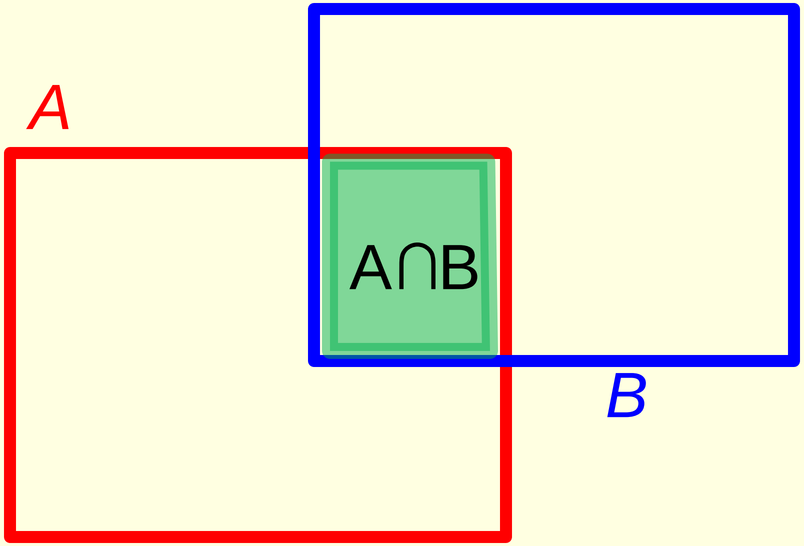
$$A \cap (A \cup B) = A$$

De Morgan's Laws:

$$(A \cup B)' = A' \cap B'$$

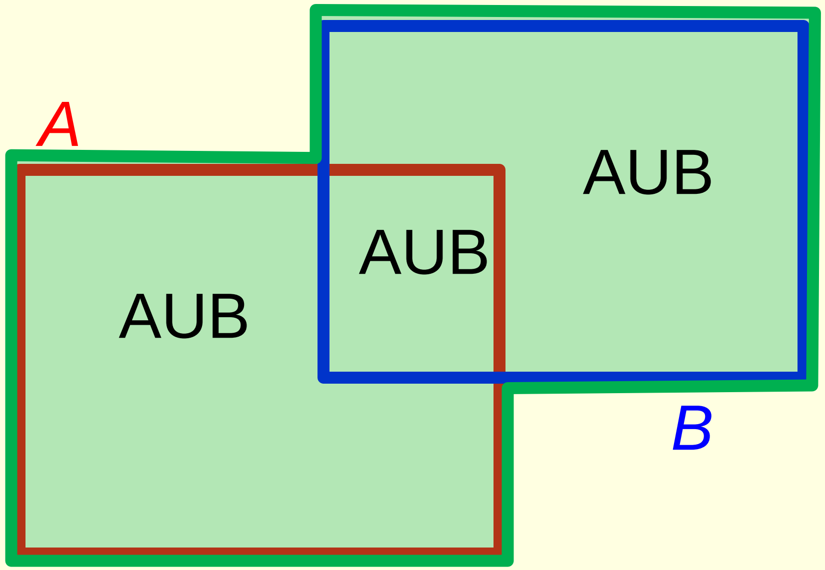
$$(A \cap B)' = A' \cup B'$$

Absorption 1: $A \cup (A \cap B) = A$



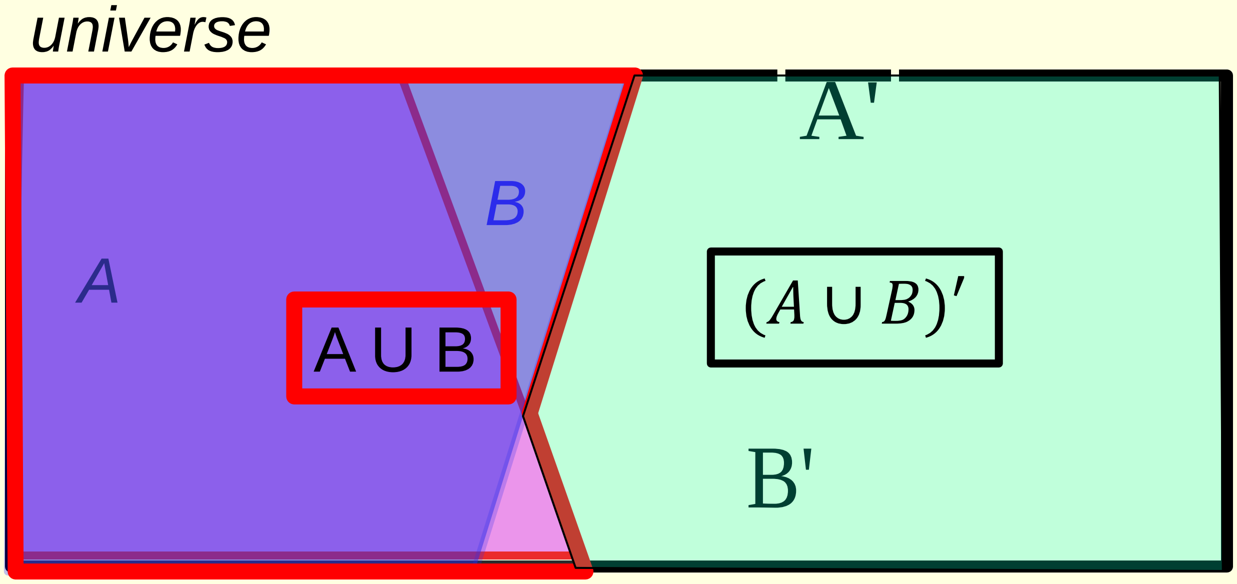
So $A \cup (A \cap B) = A$

Absorption 2 : $A \cap (A \cup B) = A$



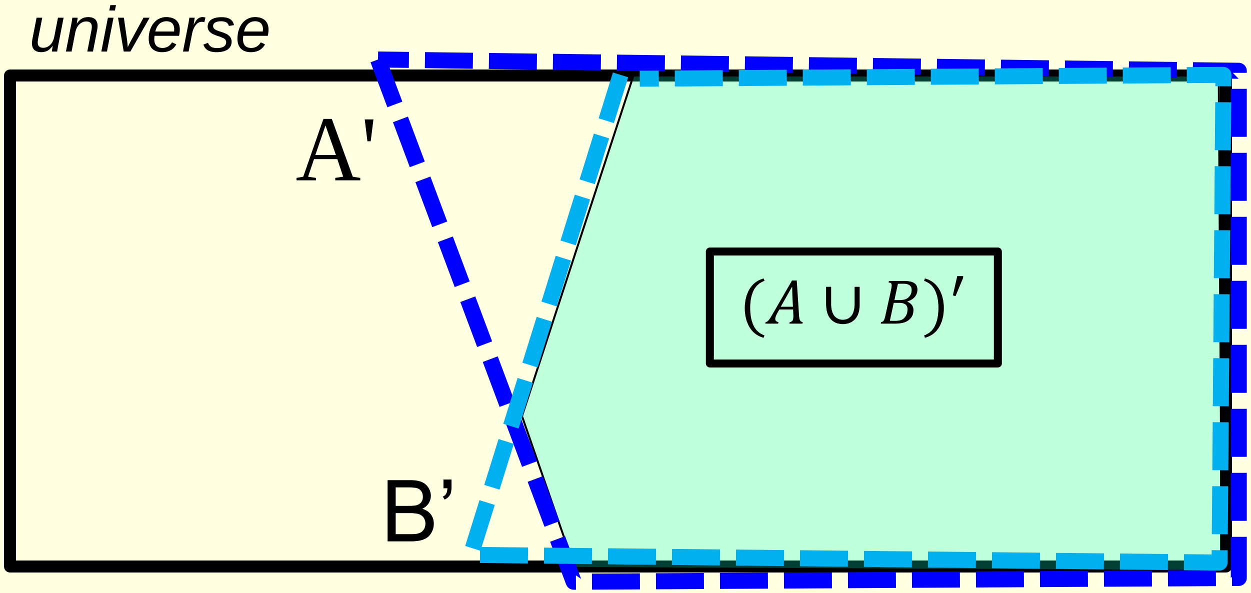
So $A \cap (A \cup B) = A$

De Morgan's Law 1: $(A \cup B)' = A' \cap B'$



So $(A \cup B)' = A' \cap B'$

De Morgan's Law 1: $(A \cup B)' = A' \cap B'$



So $(A \cup B)' = A' \cap B'$

- Properties:

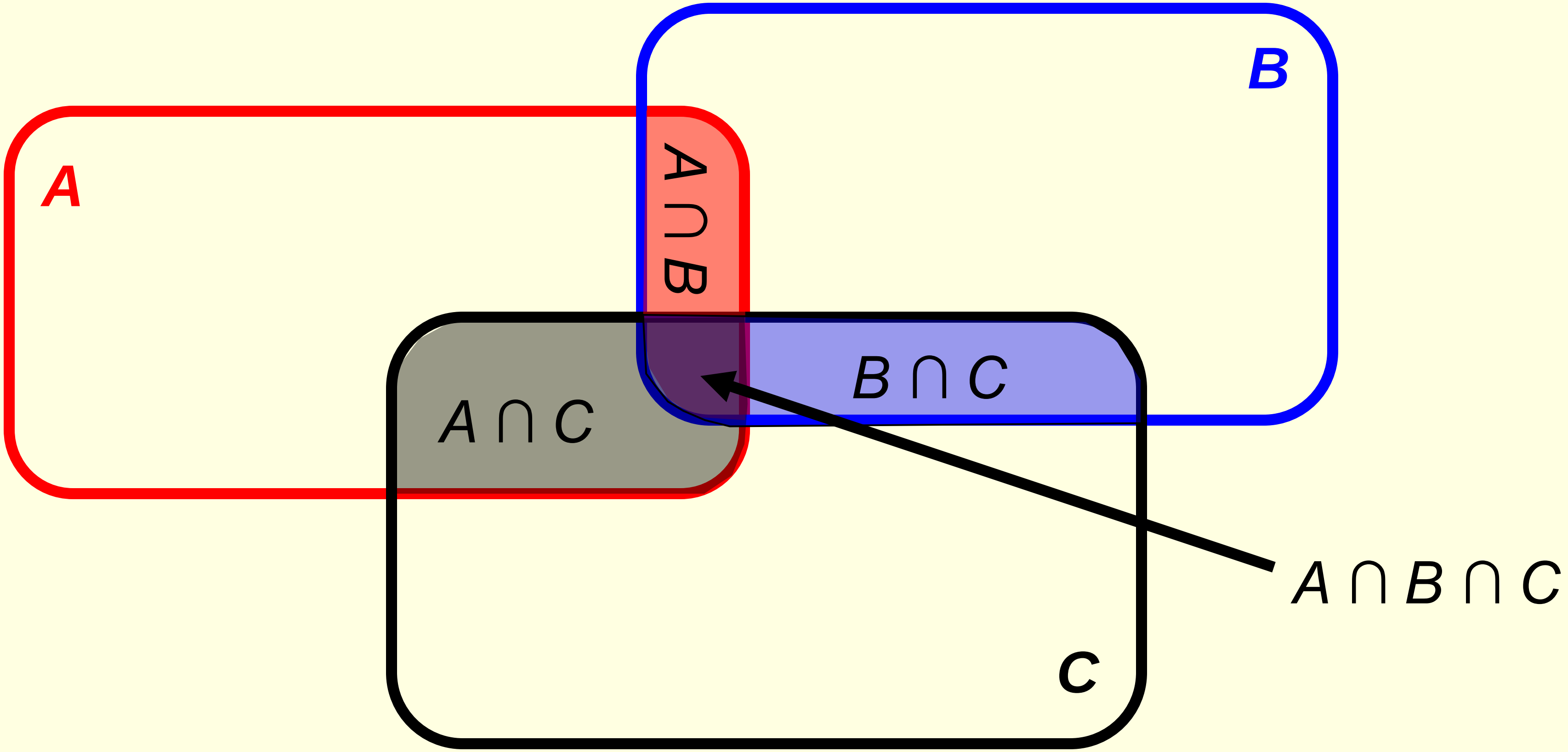
$|S|$: *cardinality of S*

Union rule : $|A \cup B| = |A| + |B| - |A \cap B|$

Difference rule : $|A - B| = |A| - |A \cap B|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Union rule : $|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$

$= |A| + |B| - |A \cap B| + |C|$

$- |(A \cap C) \cup (B \cap C)|$

Why?

$= |A| + |B| + |C| - |A \cap B|$

$- |A \cap C| - |B \cap C|$

$+ |A \cap B \cap C|$

Union rule : $|A \cup B| = |A| + |B| - |A \cap B|$

Here is why:

$|(A \cap C) \cup (B \cap C)|$

$= |A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|$

$= |A \cap C| + |B \cap C| - |A \cap B \cap C|$

Inductively defined sets: used for sets with a linear order

- To define a set *S* *inductively is to do three things:*

Basis: *Specify one or more elements of S.*

Induction: *Specify one or more rules to construct elements of S from existing elements of S.*

Closure: *Specify that no other elements are in S (always assumed).*

(Basis elements and induction rules are called ***constructors***)

Example. Find an *inductive definition* for

$$S = \{\Lambda, ac, aacc, aaaccc, \ldots\} = \{ a^n c^n \mid n \in \mathbf{N} \}.$$

Solution:

Basis: $\Lambda \in S$.

Induction: If $x \in S$ then $axc \in S$.

$$a^n c^n = a a^{n-1} c^{n-1} c$$

Example. Find an *inductive definition* for

$$S = \{ a^{n+1} b c^n \mid n \in \mathbf{N} \}.$$

Solution:

Basis: $ab \in S$.

Induction: If $x \in S$ then $axc \in S$.

$$a^{n+1} b c^n = a a^n b c^{n-1} c$$

Functions

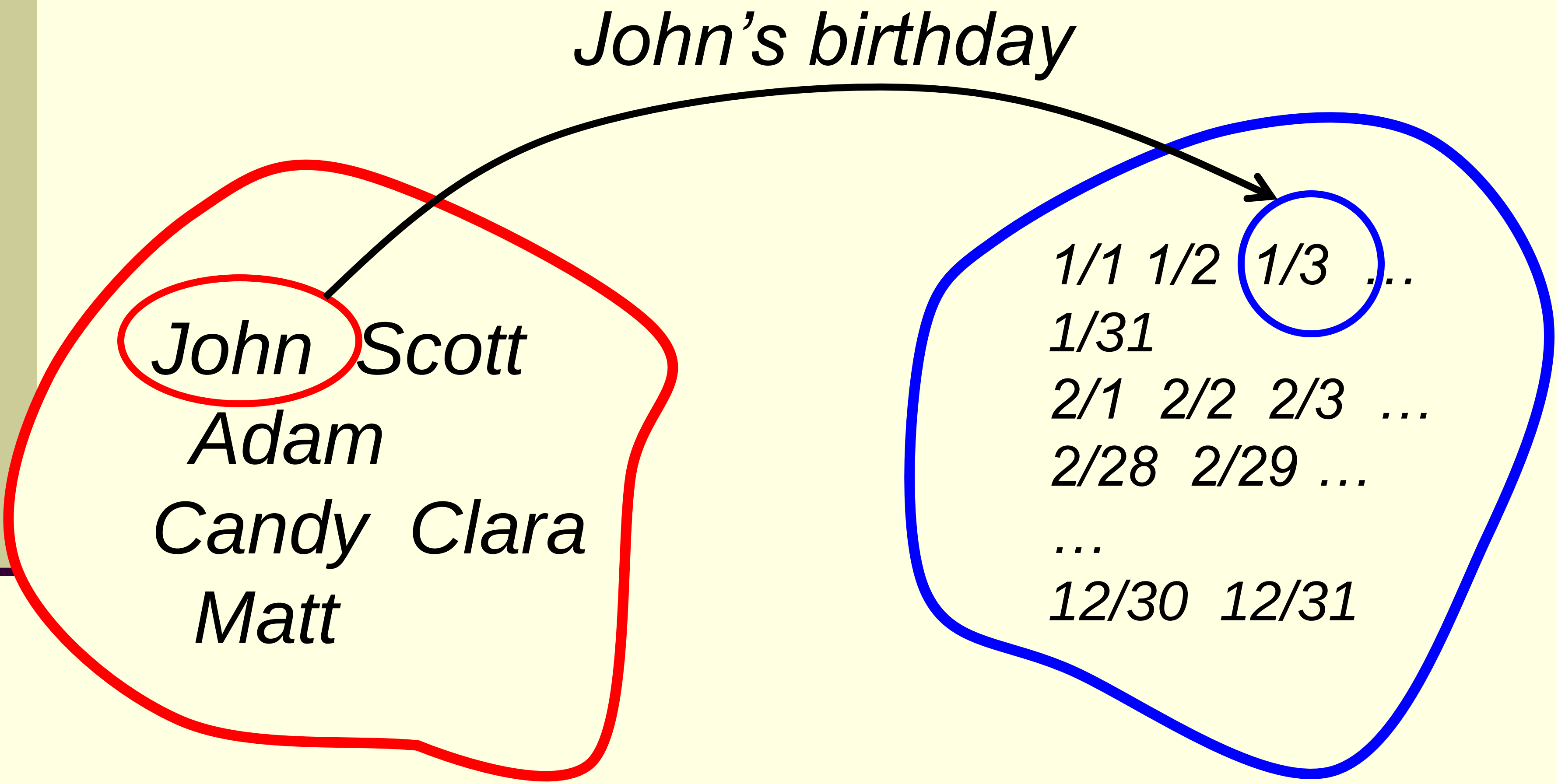
Must be precise and unique

What is a function?

a function is a way to classify/characterize things.

For instance, you can classify/characterize a group of people *<u>by their birthdays</u>.*

What is a function?



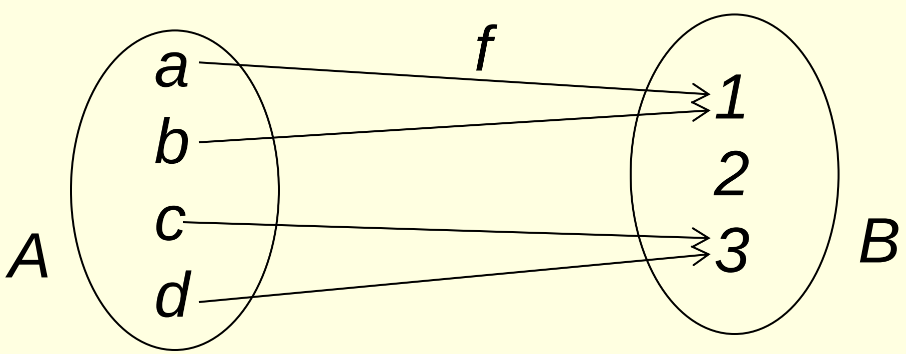
Functions

- A **function** *f* from *A* to *B* associates each element of *A* with EXACTLY one element of *B*.
- Notations: $f : A \rightarrow B$

domain (A) — *codomain* (*image*) (B)

f(a)=f(b)=1

f(c)=f(d)=3



Function?

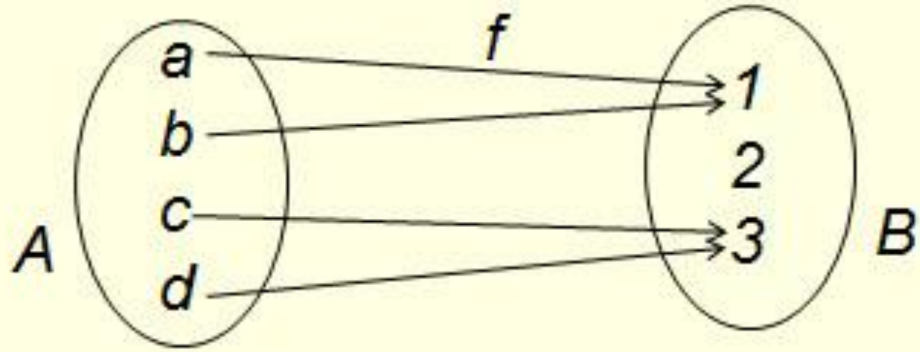
Yes

Functions

- Notations:

$f(a)=f(b)=1$

$f(c)=f(d)=3$



$range(f) = f(A) = \{f(x) \mid x \in A\} = \{1,3\}$

$f(\{a,b\}) = \{1\}$

$$f^{-1}(\{2\}) = \phi$$

$$f^{-1}(\{1,2,3\}) = \{a,b,c,d\}$$

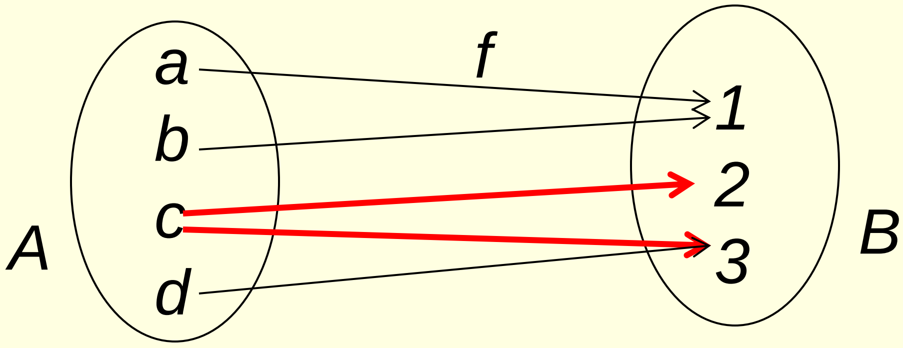
(*Inverse image of {2}*)

Functions

f(a)=f(b)=1

f(c)=f(d)=3

f(c) = 2



Function?

No

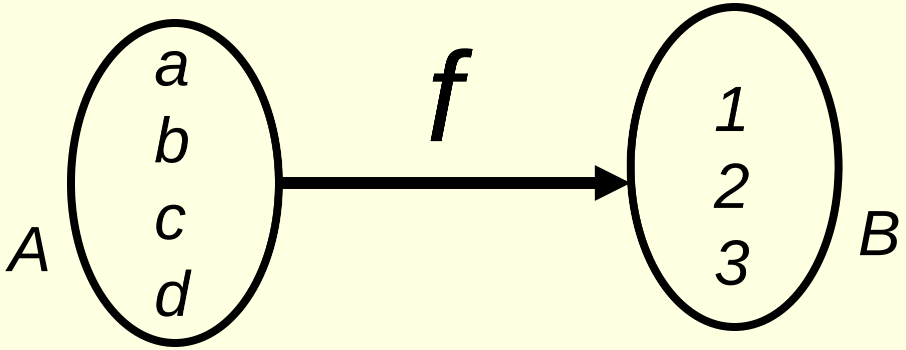
Floor and Ceiling functions:

$$\lfloor x \rfloor = floor(x)$$: *largest integer not greater than x*

$$\lceil x \rceil = ceiling(x)$$: *smallest integer not less than x*

are functions from R -> Z

Functions



How many different functions from A to B can be defined?

$$3 \times 3 \times 3 \times 3 = 3^4$$

Things you need to know about functions:

- *Composition of functions*
- *Inverse function*
- *GCD, Division algorithm, Euclid algorithm*
- *Mod functions and inverses*
- *Pigeon Hole Principle*
- *Hash functions*
- *Recursively defined functions*
- *Binary trees*

Functions

Greatest Common Divisor (gcd):

x, y : integers, not both zero

gcd(x, y) = largest integer that divides *x* and *y*

Is *gcd(x, y)* a function? ***Yes***

gcd(a,b) = gcd(b,a) = gcd(a, -b)

gcd(a,b) = gcd(b,a-bq) for some integer q

gcd(a,b) = ma+nb for some integers m and n

If d|ab and gcd(d,a) = 1, then d|b

Functions

$q = a / b$ $r = a \% b$

Division algorithm

a, b : integers, $b \mathrel{!}= 0$

there exist unique integers *q* and *r* such that

$a = bq + r, \qquad 0 <= r < |b|$

Euclid's algorithm (for finding *gcd(a,b)*)

a, b: natural integers, not both zero

```
while  (b > 0)  do {
   find q, r so that  a = bq+r   and  0<= r < b;
   a :=  b;     b := r ;
}
Output(a);
```

Functions

Examples: find *gcd(189, 33)*

$$a = q * b + r$$

$189 = \boxed{5}\ 33 + \boxed{24}$

$33 = \boxed{1}\ 24 + \boxed{9}$

$24 = \boxed{2}\ 9 + \boxed{6}$

$9 = \boxed{1}\ 6 + \boxed{3}$

$9 = \boxed{1}\ 6 + \boxed{3}$

$6 = \boxed{2}\ 3 + \boxed{0}$

$3 = \boxed{\ }\ 0 + \boxed{\ }$

Since b=0, so output a=3

Functions

mod function

a, b : integers with *b > 0*

a mod b = r if $a = bq + r$ with $0 <= r < b$

Division algorithm

How to compute q and r?

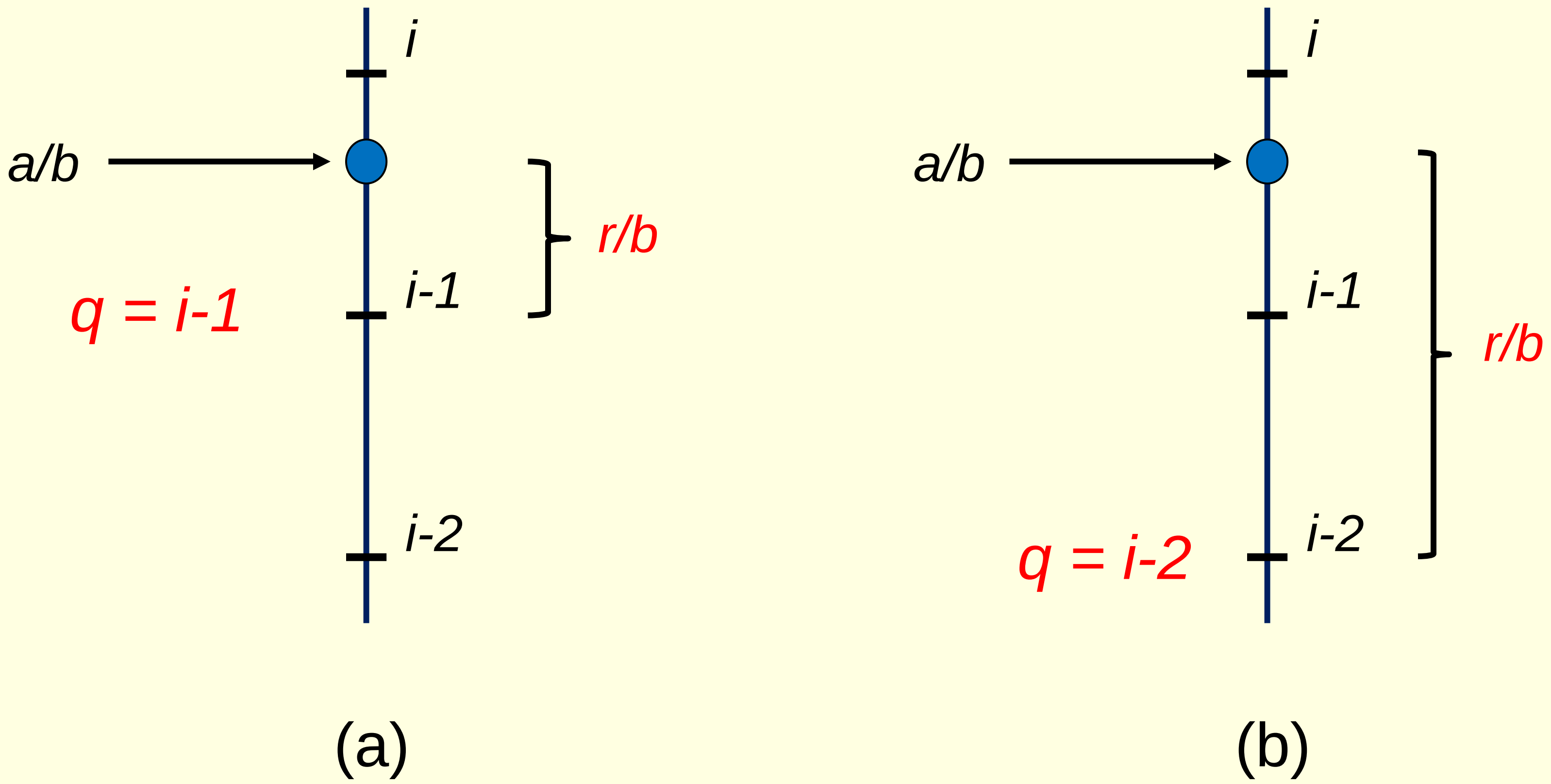
Solve the equation for $q = a/b - r/b$

Since q is an integer and $0 <= r/b < 1$

it follows that $q = \lfloor a/b \rfloor$

So we have $r = a - bq = a - b \cdot \lfloor a/b \rfloor$

$q = i\text{-}1$ or $i\text{-}2$?



Functions

mod function

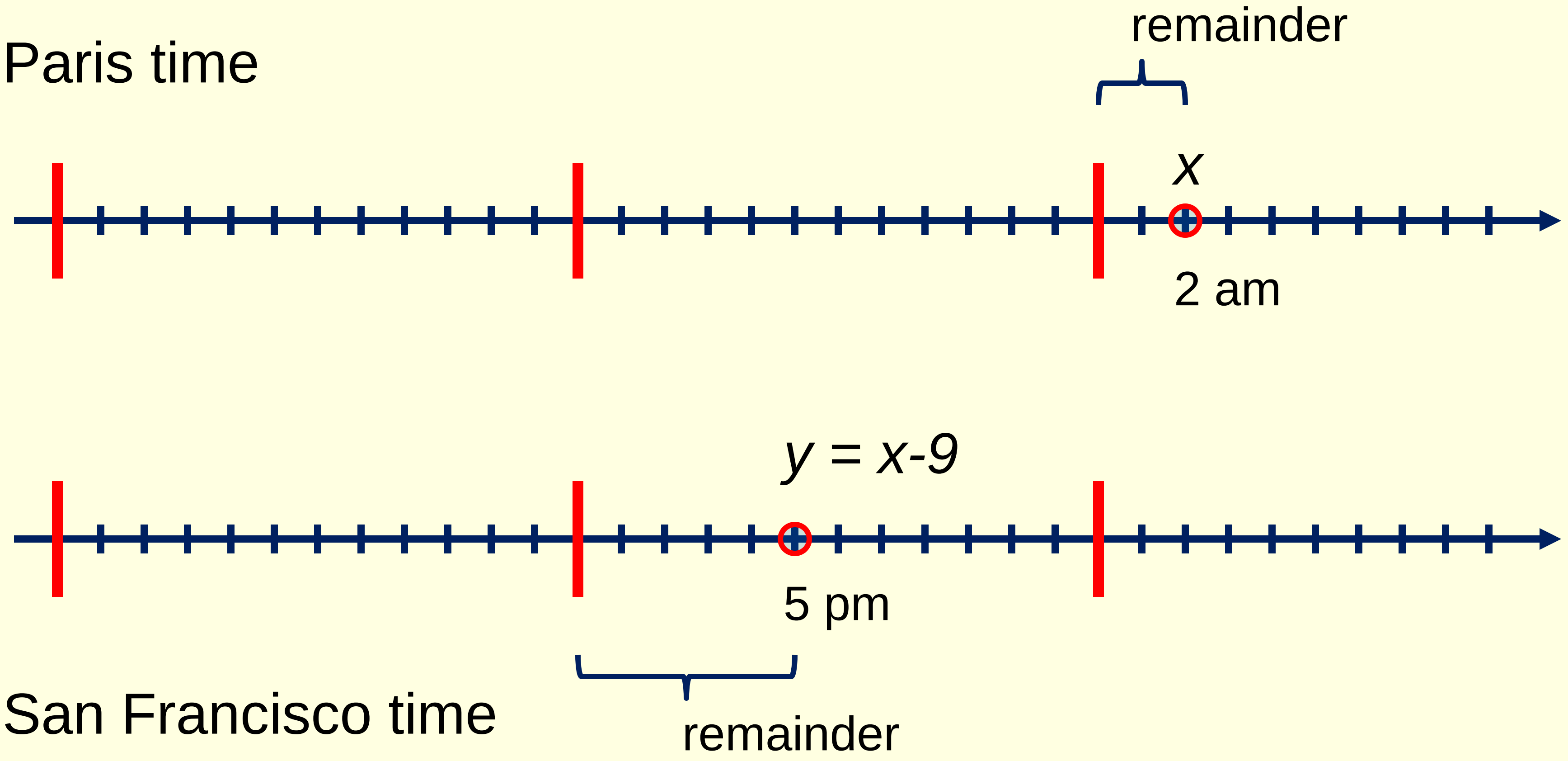
a, b : integers with *b>0*

$a \bmod b = a - b \cdot \lfloor a/b \rfloor$.

Example: It is *2am* in Paris. What time is it in San Francisco (9 hours difference)?

(12 hr clock): *(2-9) mod 12 = (-7) mod 12*

$= -7 - 12\lfloor -7/12 \rfloor = -7 - 12(-1) = 5$ pm


remainder
Paris time
x
2 am
y = x-9
5 pm
San Francisco time
remainder

Constructing Functions

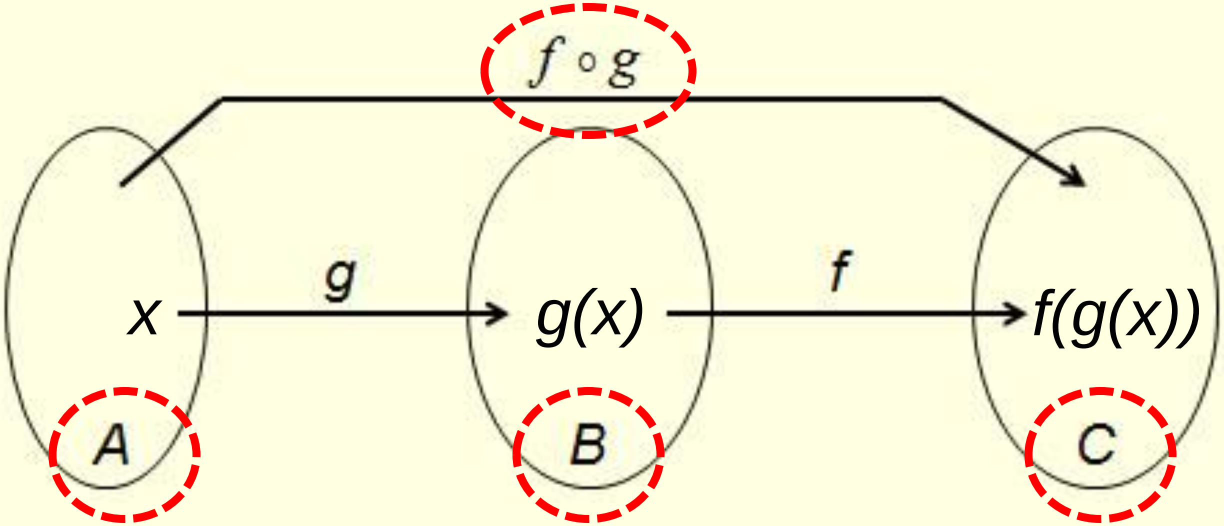
Composition:

g : A -> B and *f : B -> C*

Composition of *f* and *g* :

$$(f \circ g): A \rightarrow C$$

$$(f \circ g)(x) = f(g(x))$$



Constructing Functions

Composition:

g : A -> B and f : B -> C

Composition of f and g : $\left(f \circ g\right): A \rightarrow C$

$$\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$$

Examples:

floor ($\log_2$ 20) = floor (4.xx) = 4

ceiling ($\log_2$ 20) = ceiling (4.xx) = 5

Properties of Functions

Given: *f : A -> B*

Injective (one-to-one) : $x \neq y \Rightarrow f(x) \neq f(y)$

Surjective (onto): $\forall b \in B \quad \exists a \in A \quad such\ that \quad b = f(a)$

Bijective (one-to-one & onto): *injective + surjective*

Inverse: *If f is a bijection, then the inverse of f,* f^{-1} *, exists and is defined by* $f^{-1}(b) = a$ *iff* $f(a)=b$

Properties of Functions

Injective (one-to-one) : $x \neq y \Rightarrow f(x) \neq f(y)$

Example: $f: N_8 \rightarrow N$ where $N_n = \{0,1,\ldots,n-1\}$

$f(x) = 2x \bmod 8$

Is f injective? (f(0) =? f(4) = ?)

NO

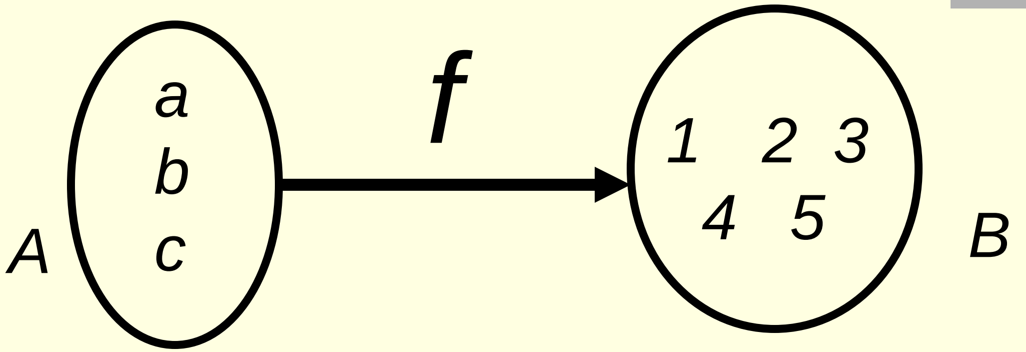
Example: f : Z -> N

$f(x) = x^2$

Is f injective? (f(2) = ? f(-2) = ?)

NO

One-to-one Functions



How many different one-to-one functions from A to B can be defined?

$$5 \times 4 \times 3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$$

Properties of Functions

Surjective (onto): $\forall b \in B \quad \exists a \in A \quad such\ that \quad b = f(a)$

Example: $f : Z \rightarrow N$

$$f(x) = x^2$$

Is f surjective?

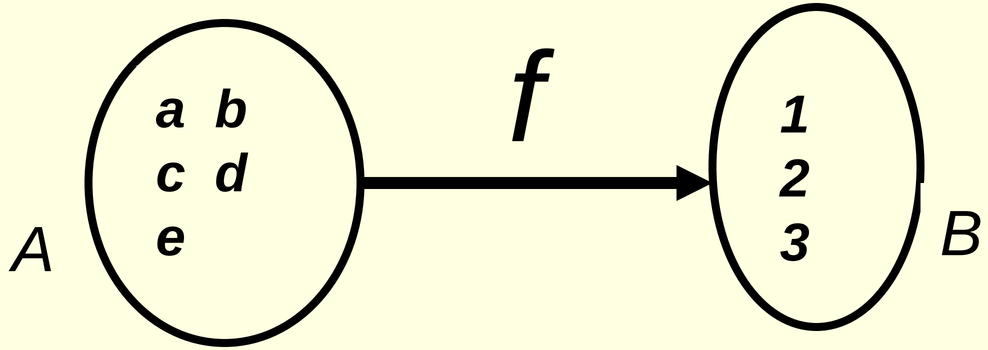
NO

Example: $f : Z \rightarrow N$

$$f(x) = |\, x \,|$$

f surjective but not injective

Onto Functions

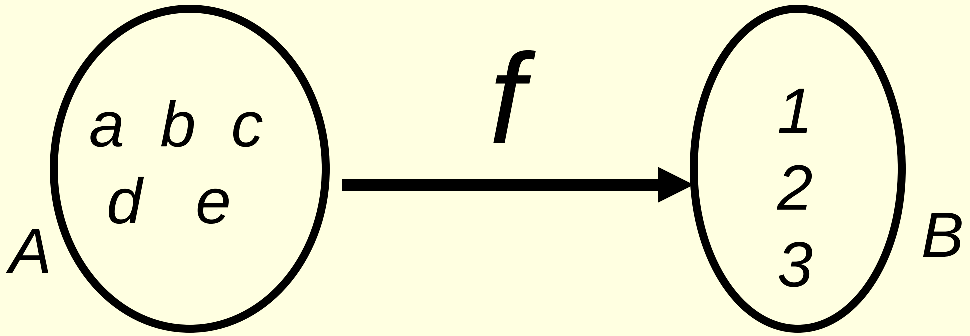


How many different onto functions from A to B can be defined?

$$3^5 - F_1 - F_2$$

where F_i $(i = 1, 2)$ $is\ the$ number of different functions from A to B with each image set having i elements only.

Onto Functions

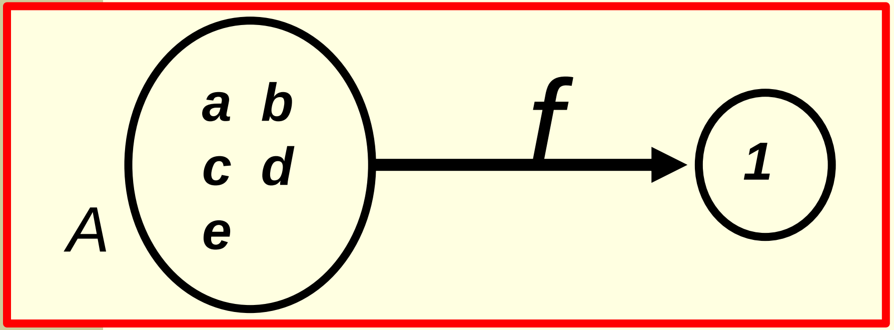


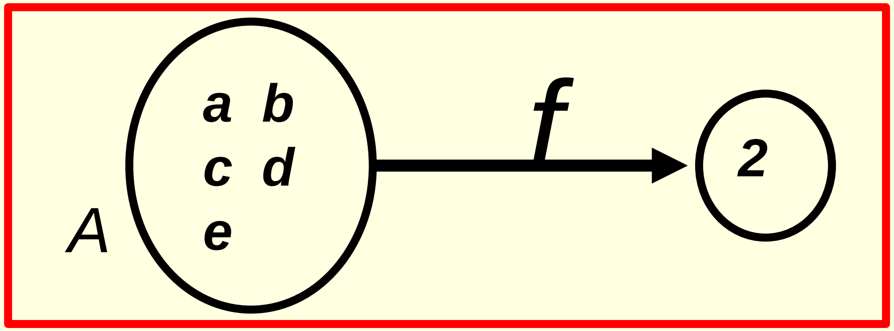
$$F_1 = 3; \quad F_2 = (2^5 - 2)\binom{3}{2}$$

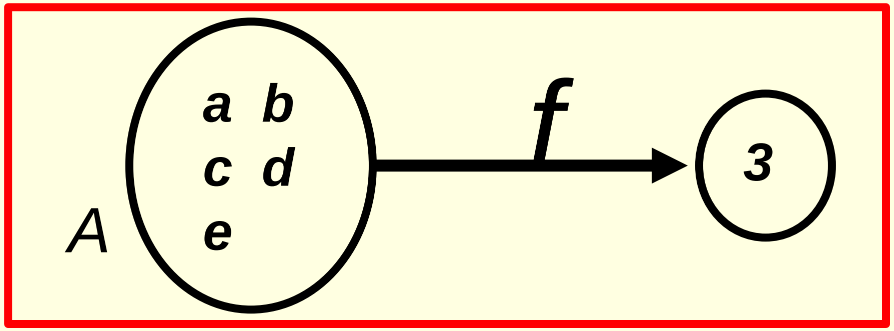
Hence, the number of different onto functions from A to B = $3^5 - 3 - (2^5 - 2)\binom{3}{2}$

Onto Functions

$F_1 = 3$ *Why?*



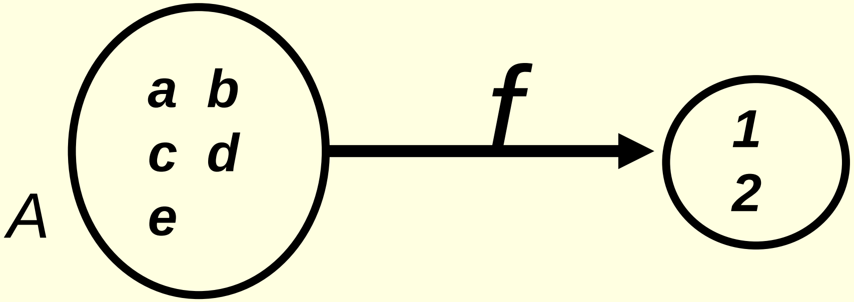




Onto Functions

$$F_2 = (2^5 - 2)\binom{3}{2}$$ *Why?*

For a 2-element subset of B, the number of onto functions from A to that subset is: $2^5 - 2$



And B has $\binom{3}{2}$ *2-element subsets.*

So $F_2 = (2^5 - 2)\binom{3}{2}$

Properties of Functions

Bijective (one-to-one & onto):

Example: $f : (0, 1) \to (2, 5)$ *defined by* $f(x) = 3x + 2$ *is a bijection*

Proof:

Example: $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = x^3$

Is f bijective ? ***Yes***

Functions

$$f : A \to B$$

$|A| = m \qquad |B| = n$

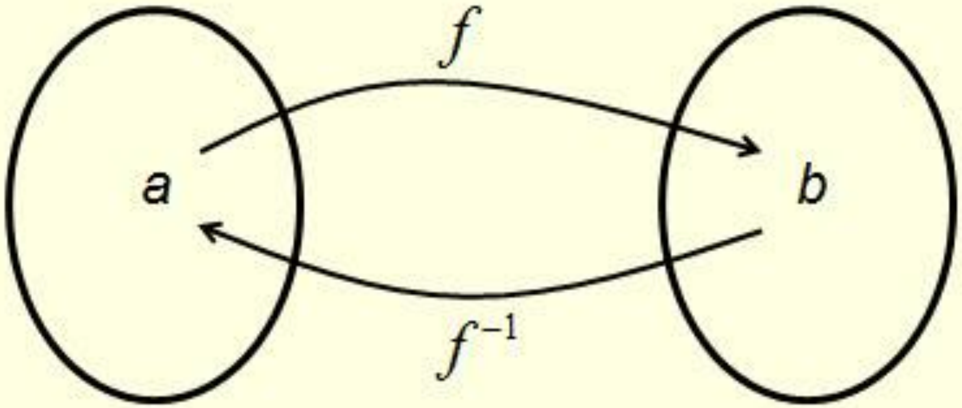
(1) How many different functions can be defined from A to B?

(2) If $m \le n$ then how many different one-to-one functions can be defined from A to B?

(3) If $m \ge n$ then how many different onto functions can be defined from A to B?

Properties of Functions

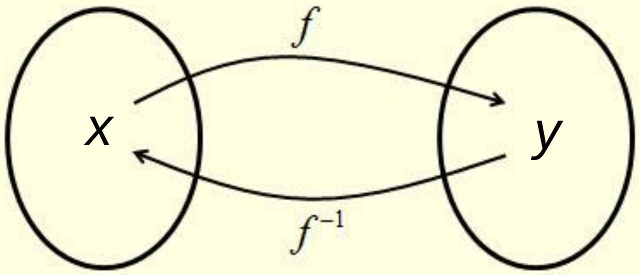
Inverse: *If f is a bijection, then the inverse of f,* f^{-1}*, exists and is defined by* $f^{-1}(b) = a$ *iff* $f(a) = b$



Hence, $f^{-1}(f(a)) = ?$ $f(f^{-1}(b)) = ?$

$$f^{-1}(f(a)) = a \qquad f(f^{-1}(b)) = b$$

Properties of Functions



Example: $f : (0, 1) \rightarrow (2, 5)$ defined by $f(x) = 3x + 2$ is a bijection

$$f^{-1}(y) = ?$$

Note that $f(f^{-1}(y)) = y$

On the other hand, if we think of $f^{-1}(y)$ as an x, then by definition, we have

$$f(f^{-1}(y)) = 3f^{-1}(y) + 2 = y$$

So, $f^{-1}(y) = (y - 2)/3$

Properties of Functions

$N_5 = \{0, 1, 2, 3, 4\}$

$f(N_5) = \{1, 0, 4, 3, 2\}$

Example: $f : N_5 \to N_5$ defined by $f(x) = (4x+1) \bmod 5$ is a bijection

$f^{-1} = ?$

Theorem (mod and inverses)

Let $n > 1$ and $f : N_n \to N_n$ be defined by $f(x) = (ax+b) \bmod n$. Then

- f is bijective iff $\gcd(a, n) = 1$
- If so, then $f^{-1}(x) = (kx + c) \bmod n$ where $f(c)=0$ and $1=ak+nm$

Why should we take '*c*' and '*k*' such that $f(c) = 0$ and $1 = ak + nm$?

To ensure that $f\left(f^{-1}(y)\right) = y$

$$f\left(f^{-1}(y)\right) = f(ky + c) = a(ky + c) + b$$

$$= aky + ac + b$$

$$= aky + qn \quad for\ some\ q\ \epsilon Z$$

$$= (1 - nm)y + qn$$

$$= y - nmy + qn = y \bmod n$$

Properties of Functions

Example: $f : N_5 \to N_5$ defined by $f(x) = (4x + 1) \bmod 5$ is a bijection

$f^{-1} = ?$

Since gcd(4, 5) = 1, the theorem says that f is a bijection.

First, find a 'c' such that $f(c) = 0$. e.g., $f(1) = 0$.

Then use Euclid's algorithm to verify that 1 = gcd(4, 5) and work backwards through the equations to find that

$1 = 4(-1) + 5(1)$. So $k = -1$.

Thus $f^{-1}(x) = (-x + 1) \bmod 5$. $= (4x + 1) \bmod 5$

Properties of Functions

Find *gcd(5, 4)*

Find $1 = 4 \cdot (-1) + 5 \cdot 1$

$a = q * b + r$

5	$=$	1	4	$+$	1	
4	$=$	4	1	$+$	0	
1	$=$		0	$+$		

$1 = 5 - 1 \cdot 4$

$1 = 5 \cdot (1) + 4 \cdot (-1)$

Since b=0, so output a=1

Properties of Functions

Find $gcd(23, 4)$

Find $1 = 4 \cdot (6) + 23 \cdot (-1)$

$$23 = \boxed{5}\ 4 + \boxed{3}$$

$$4 = \boxed{1}\ 3 + \boxed{1}$$

$$3 = \boxed{3}\ 1 + \boxed{0}$$

$$1 = \boxed{}\ 0 + \boxed{}$$

$$1 = 4 - 1 \cdot 3$$

$$1 = 4 + (23 - 5 \cdot 4) \cdot (-1)$$

$$1 = 4 \cdot 6 + 23 \cdot (-1)$$

Since $b=0$, so output $a=1$

Number of elements in the domain

Number of elements in the codomain

Functions

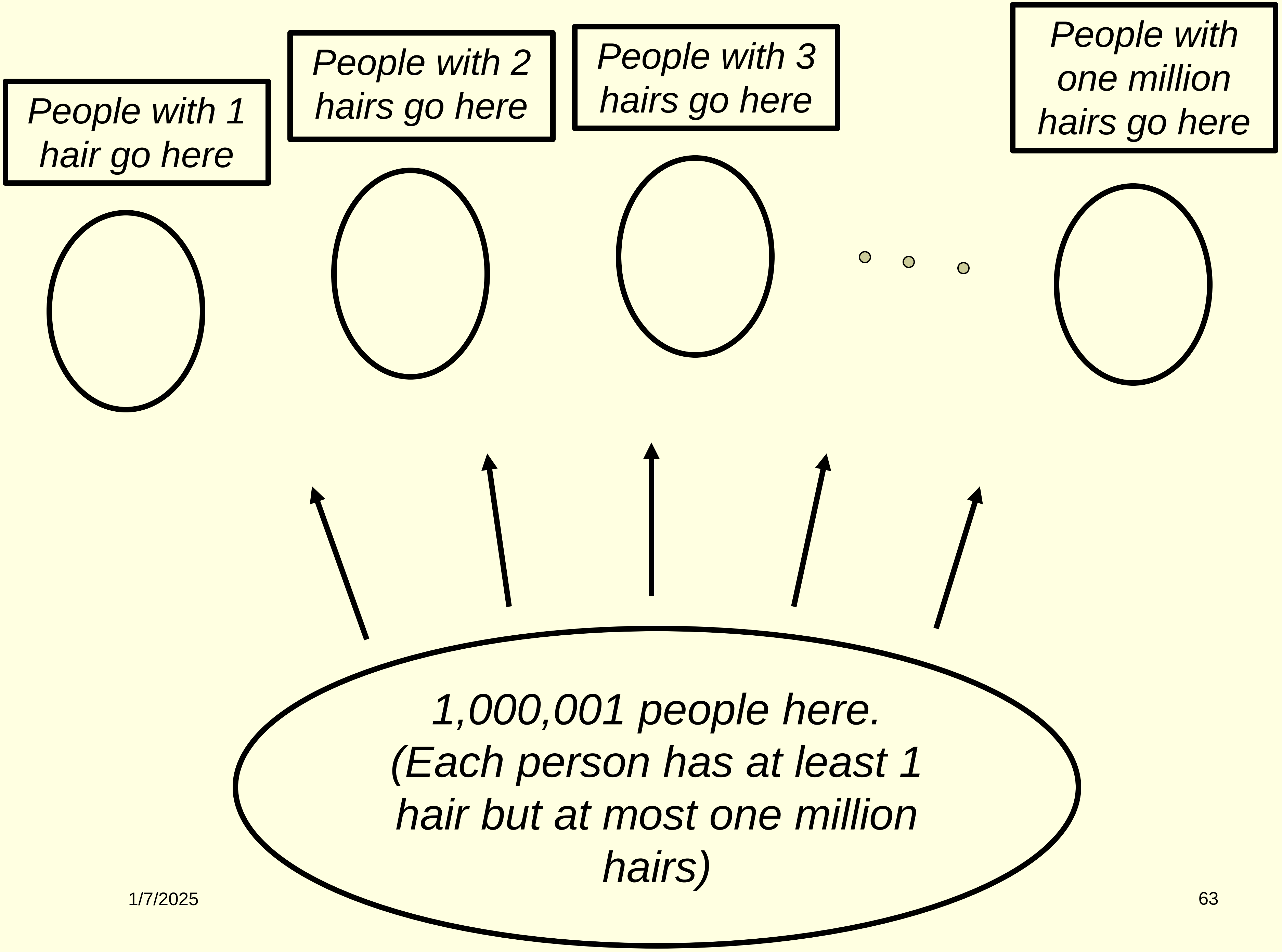
- **P on Hole Principle** *If m pig s are put into n ho and m > n, then one hole has two more pigeons. (or A and B are finite sets with | A | > |, then there are no ections from A to B)*

Exampl . *In Mexico City there are two pe le with the same number f hairs on their heads (assumptic : everyone has less thar 10 million hairs on their head and the population of Mexico City is more than 10 million).*

Number of pigeons=?   *Number of holes = ?*

Population of Mexico City

Number of people with different number of hairs

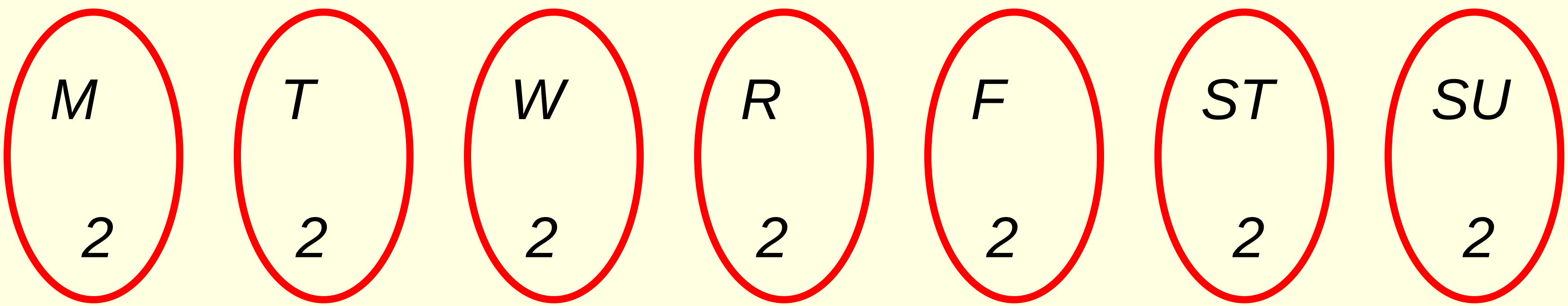

People with 1 hair go here
People with 2 hairs go here
People with 3 hairs go here
People with one million hairs go here
1,000,001 people here.
(Each person has at least 1 hair but at most one million hairs)

Properties of Functions

Example. *How many people are needed in a group to say that three were born on the same day of the week?*

Solution:

would 14 people work?



would 15 people work?

Properties of Functions: hashing

Hash Functions use keys to look up information in a table, but without searching, so we can cut operation time from O(n) to O(1).

Items are stored into a table (called a hash table) based on the value of a search key (e.g., birthday).

If collisions occur, then a second key (e.g., last name) is used.

The idea is to use the keys to jump directly to the entry where the information is stored without any searching.

Properties of Functions: hashing

The operation time is O(1)

How to find T. Smith's information?

Get T. Smith's birthday : March 1

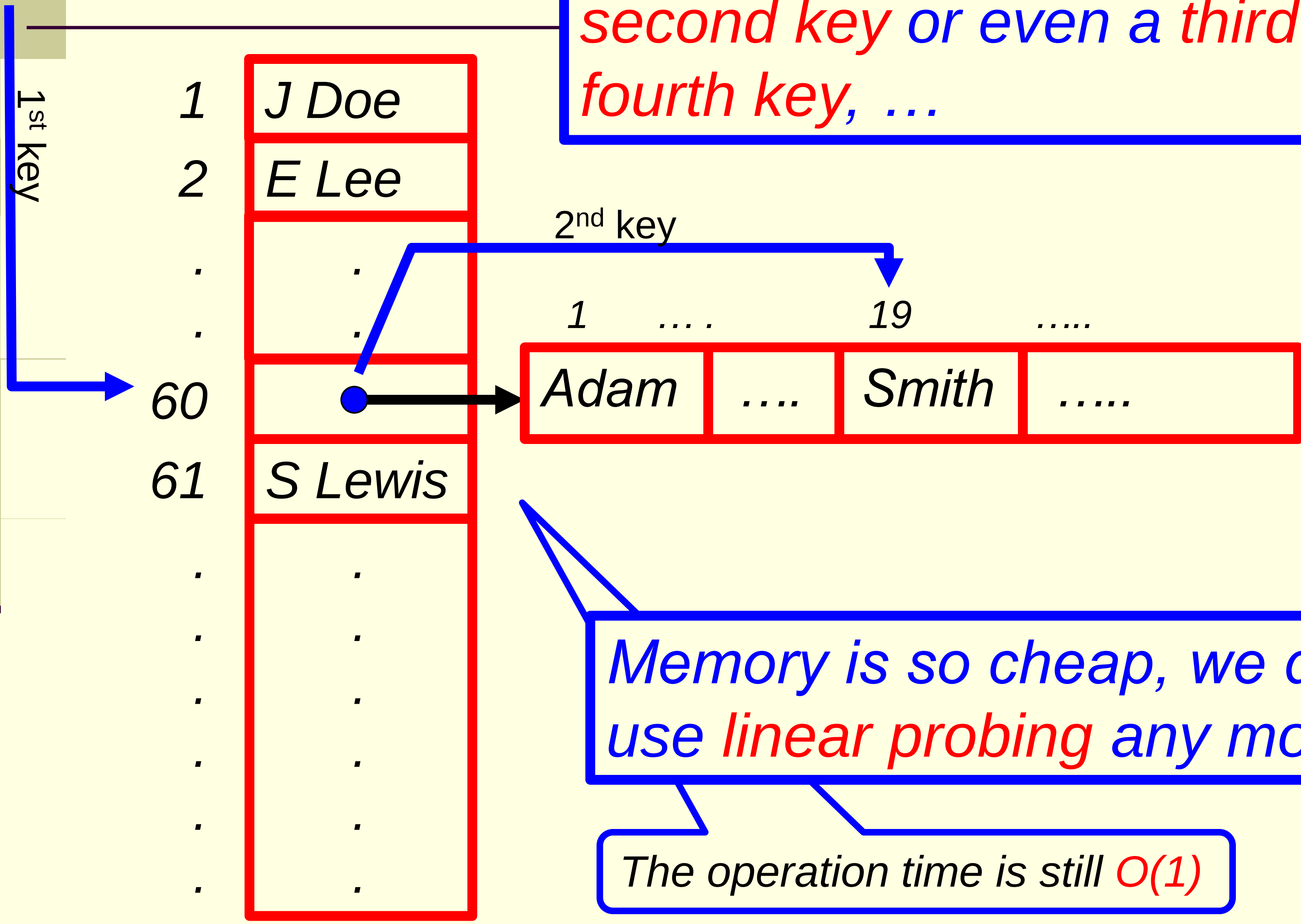
Compute the index:
31+28+1 = 60

So go to entry 60 to get T. Smith's information

1	J Doe
2	E Lee
.	.
.	.
60	T Smith
61	S Lewis
.	.
.	.
.	.
.	.
.	.
.	.

Properties of Functions: hashing

To avoid collision problem, use a second key or even a third key, a fourth key, …



Memory is so cheap, we don't use linear probing any more.

The operation time is still O(1)

String Algebra

Given an alphabet A={a, b, c, d, e}

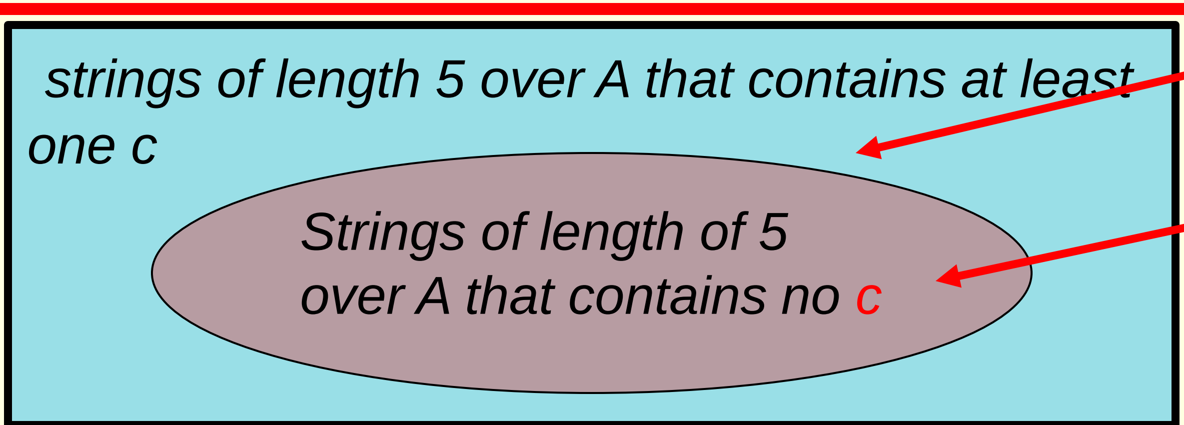
1. what is the number of strings of length 5 over A that ends in *a* or *b*?

5	5	5	5	a/b

$(5^4 * 2)$

2. what is the number of strings of length 5 over A that contains at least one c?

Strings of length 5 over A



$(5^5 - 4^5)$

4^5

More on Functions

Recursiveness does not save computation time, it only makes your code more compact

Recursively Defined Functions

Function ƒ is recursively defined : at least one ƒ(x) is defined in terms of another ƒ(y), where x ≠ y.

```
sum = 0;
for (i=1; i<=1000; i++)   sum = sum + i;
```

f(0) = 0;
f(n) = f(n-1) + rule(s) on n

Technique (when argument domain is inductively defined)

1. *Specify a value ƒ(x) for each basis element x of S.*
2. *Specify rules that,*
   for each inductively defined element x in S,
   define ƒ(x) in terms of previously defined values of ƒ.

More on Functions

One way to write the code:

```
sum = 0;
sum = sum + 1;
sum = sum + 2;
sum = sum + 3;
sum = sum + 4;
sum = sum + 5;
        ⋮
sum = sum + 1,000,000;
```

Computation time is the same

A better way to write the code:

```
sum = 0;
for (i=1; i<=1000; i++)   sum = sum + i;
```

Binary Relations

Relation is a way to partition a set

A ***binary relation*** R over a set A is a subset of $A \times A$.

If $(x, y) \in R$ we also write xRy.

Example. Binary relations over $A = \{0, 1\}$:

$\emptyset$, $A \times A$, **eq** $= \{(0, 0), (1, 1)\}$, **less** $= \{(0, 1)\}$.

Definitions: Let R be a binary relation over a set A.

- R is ***reflexive*** : xRx for all $x \in A$.
- R is ***symmetric*** : $xRy \Rightarrow yRx$ for all $x, y \in A$.
- R is ***transitive*** : $xRy, yRz \Rightarrow xRz$ for all $x, y, z \in A$.

Binary Relations

Composition: If *R and S are binary relations, then composition of R and S is*

$R \circ S = \{(x, z) \mid xRy \text{ and } ySz \text{ for some } y\}.$

$(x, y) \circ (y, z) = (x, z)$

***Example* (*digraph representations*).** *Let R = {(a, b), (b, a), (b, c)} over A = {a, b, c}. Then R,* $R^2 = R \circ R$, *and* $R^3 = R^2 \circ R$ *can be represented by directed graphs:*

$$R = \{ (a, b), (b, a), (b, c) \}$$

$$R^2 = \{ (a, a), (b, b), (a, c) \}$$

$$R^3 = \{ (a, b), (b, a), (b, c) \}$$

Equivalence Relations

Provides a way to partition the domain set

A binary relation is an ***equivalence relation*** *if it has the three properties:* **reflexive**, **symmetric**, and **transitive** (RST).

***Examples**. a.* Equality on any set.

b. x ~ y iff | x | = | y | over the set of strings {a, b, c}*.

c. x ~ y iff x and y have the same birthday over the set of people.

Quiz. *Which of the relations are RST?*

a. xRy iff $x \leq y$ or $x > y$ over Z.

b. xRy iff $|x - y| \leq 2$ over Z. *(Not transitive)*

c. xRy iff x and y are both even over Z. *(Not reflexive)*

Answers. Yes, No, No.

Equivalence Relations

Equivalence Classes: If R is RST over A, then for each $a \in A$ the *equivalence class* of a, denoted [a], is the set [a] = {x | xRa}.

Property: For every pair $a, b \in A$ we have either [a] = [b] or
[a] ∩ [b] = ∅.

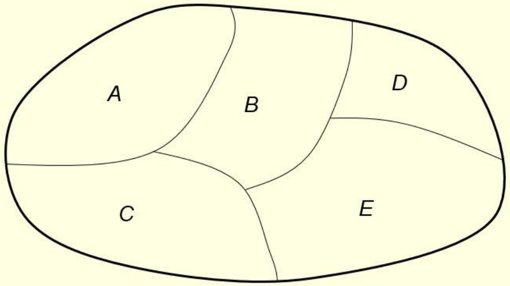
Example. Suppose x ~ y iff x mod 3 = y mod 3 over **N.** Then the equivalence classes are,

$[0] = \{0, 3, 6, \ldots\} = \{3k \mid k \in \mathbf{N}\}$

$[1] = \{1, 4, 7, \ldots\} = \{3k + 1 \mid k \in \mathbf{N}\}$

$[2] = \{2, 5, 8, \ldots\} = \{3k + 2 \mid k \in \mathbf{N}\}$.

Equivalence Relations



A **Partition** of a set is a collection of nonempty disjoint subsets whose union is the set.

Example. From the previous example, the sets [0], [1], [2] form a partition of ***N*.**

Theorem (RSTs and Partitions). Let *A* be a set. Then the following statements are true.

1. Equivalence classes of any RST over *A* form a partition of *A*.
2. Any partition of *A* yields an RST over *A*, where the sets of the partition act as the equivalence classes.

Equivalence Relations

Example. *Let* $x \sim y$ *iff* $x \bmod 2 = y \bmod 2$ *over* $\mathbf{Z}$. Then ~ is an RST with equivalence classes *[0]*, the evens, and *[1]*, the odds. Also *{[0], [1]}* is a partition of **Z.**

Example. ***R*** can be partitioned into the set of half-open intervals $\{(n, n + 1] \mid n \in Z\}$. Then we have an RST ~ over ***R*,** where $x \sim y$ iff $x, y \in (n, n + 1]$ for some $n \in Z$.

Refinements of Partitions. If *P* and *Q* are partitions of a set *S*, then *P* is a refinement of *Q* if every $A \in P$ is a subset of some $B \in Q$.

Equivalence Relations

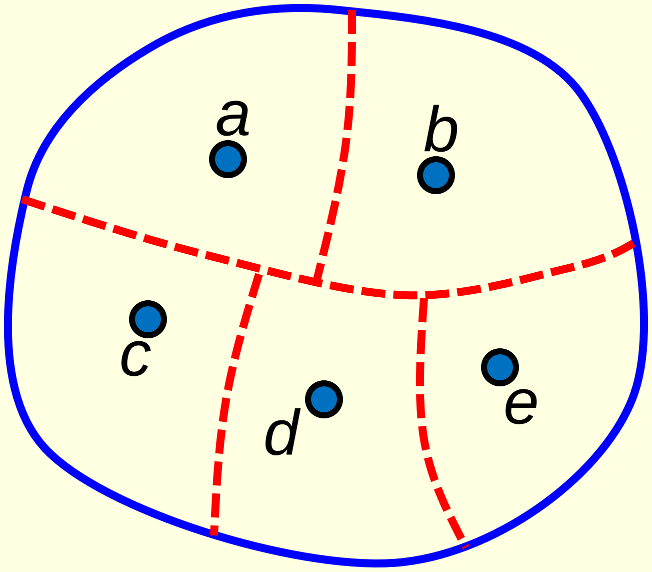
Example. Let $S = \{a, b, c, d, e\}$ and consider the following four partitions of S.

$P1 = \{\{a, b, c, d, e\}\}$,

$P2 = \{\{a, b\}, \{c, d, e\}\}$,

$P3 = \{\{a\}, \{b\}, \{c\}, \{d, e\}\}$,

$P4 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$.



Each *Pi is a refinement of Pi–1.*

P1 is the "coarsest" and P4 is the "finest".

End of Preliminaries