CS375: Logic and Theory of Computing

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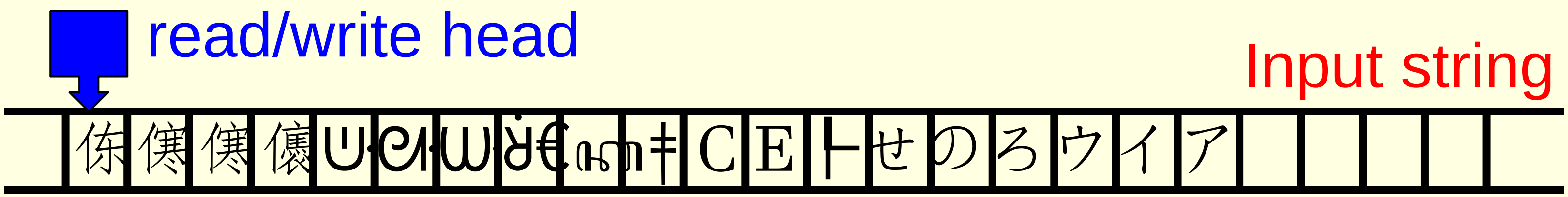
- **Weeks 12-13: Propositional Logic (Chapter 6), Predicate Logic (Chapter 7), Computational Logic (Chapter 9), Algebraic Structures (Chapter 10)**

Regular Languages & Finite Automata

- Regular Languages

Goal:

Try to answer the question: "can a machine recognize a language?"



If the answer is YES, then what kind of languages can be recognized by what kind of machines ?

A smart machine

Remember these:

First, the language must be a DIGITAL language (Otherwise, the language has to be DIGITIZED).

Second, the machine must be a DIGITAL machine, i.e., the machine must use a digital process to read/execute the language.

No digital process, No computer!

Regular Languag

A description, not a representation.
*We need something like: a*bba**

Problem:

Suppose the input strings strings over the alphabet *{a, b}* that conta s exactly one substring *bb*, i.e., the strings are of the form

xbby

where *x*, *y*: strings over *{a, b}* that do not contain *bb*, *x* does not end in *b*, and *y* does not begin with *b*.

Question: how to **represent** the strings formally?

Remember these:

Description : a set of statements

Representation : a form that carries a specific internal structure

So, is 'xbby' a representation?

Yes, but not precise enough. Why?

Regular Languages:

set of strings over A

A ***regular language*** over alphabet *A* is a language constructed by the following rules:

Means you can have single-letter words

- $\emptyset$ and $\{\Lambda\}$ are regular languages.
- *$\{a\}$ is a regular language for all $a \in A$.*
- If *M* and *N* are regular languages, then so are $M \cup N$, MN, and M^*.

Means you can have multiple-word/sentence languages

Means you can have multiple-letter words and sentences

Means you can have everything above

closure of M: all possible concatenations of strings from M

Regular Languages:

set of strings over A

A ***regular language*** over alphabet A is a language constructed by the following rules:

Basis

- $\emptyset$ and $\{\Lambda\}$ are regular languages.
- $\{a\}$ *is a regular language for all* $a \in A$.

- If M and N are regular languages, then so are $M \cup N$, MN, and M^*.

Induction

Why do we need an empty set and an empty string?

A set with three elements has how many subsets?

Eight (one of them is the empty set)

A string with three letters has how many substrings?

Eight (one of them is the empty string)

Regular Languages:

Example. Regular languages over $A = \{a, b\}$:

∅, {Λ}, {*a*}, {b}, {a, b}, {ab}, {a}*={Λ, a, aa, aaa, …}

A ***regular expression*** over A is an expression constructed by the following rules:

representation

- ∅ and Λ are regular expressions.
- a is a regular expression for all $a \in A$.
- If *R* and *S* are regular expressions, then so are

  (R), *R + S*, *R·S*, and *R**.

do R first

R or S

Remember these:

Language = **set** of strings

Expression

= **(format)** representation

= **general** representation

= **symbolic** representation

Set vs Expression

The set of polynomials in the variable x

= $\{2 + 3x - 11x^2,\ 49 + 6x^3 + 5x^7, \ldots \}$

Or

= $\{ f(x) \mid f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ *for some non-negative integer* n *and* $a_0, a_1, a_2, \ldots, a_n$ *are real numbers* $\}$

Set vs **Expression**

f(x) + g(x) = ?   f(x) g(x) = ?   f(x) = ?*

You can use real polynomials to show the addition, multiplication, subtraction … of polynomials.

But it is more general to use format representation to explain the addition, multiplication, subtraction … process.

Why do we want to study regular expressions?

Because dealing with languages (sets) directly is a very time consuming process.

Note that most of the practical regular languages are infinite sets. Dealing with infinite sets directly means we have to list those sets explicitly, a very time consuming (and actually impossible) process.

On the other hand, if we deal with regular expressions of regular languages instead, we only have to list a few representations (format descriptions), a much simpler process.

Regular Languages:

Another way to understand expression:

(1) Φ is a regular expression of the empty language.

(2) Λ is a regular expression of {Λ}.

(3) For any symbol a, a is a regular expression of {a}.

(4) If R_A and R_B are regular expressions of languages A and B, then R_A+R_B is a regular expression of A U B, R_AR_B is a regular expression of AB, and R_A* is a regular expression of A*.

Regular Languages:

Not a regular expression of {0}{1}

Examples:

- **011** : regular expression of {0}{1}{1}.
- **0+1** : regular expression of {0,1}.
- **(0+1)*** : regular expression of {0,1}*.
- Remark: **(0+1)** is also considered a regular expression of {0, 1}.

Regular Languages:

The hierarchy in the absence of parentheses is:

*, $\cdot$, +

$$a + b \cdot a^* = (a + (b \cdot (a)^*)) \equiv a + ba^*$$

Juxtaposition will be used in place of $\cdot$.

Example. The following expressions are a sampling of the regular expressions over A = {a, b}:

∅, Λ, a, b, ab, a + ab, (a + b).*

$$\begin{cases} \Lambda : \text{expression} \\ \{\Lambda\} : \text{language} \end{cases}$$

Regular Languages:

expression set expression element

Regular expressions represent regular languages by the following correspondence, where $L(R)$ denotes the regular language of the regular expression R.

$L(\emptyset) = \emptyset$,

$L(\Lambda) = \{\Lambda\}$,

$L(a) = \{a\}$ for all $a \in A$,

$L(R + S) = L(R) \cup L(S)$,

$L(RS) = L(R)L(S)$,

$L(R^*) = L(R)^*$

Regular Languages:

Example. $L(ab + a^*)$ represents the following regular language:

$$L(ab + a^*) = L(ab) \cup L(a^*) = L(a)L(b) \cup L(a)^* = \{a\}\{b\} \cup \{a\}^* = \{ab\} \cup \{\Lambda, a, aa, aaa, \ldots, a^n, \ldots\} = \{ab, \Lambda, a, aa, aaa, \ldots, a^n, \ldots\}.$$

$L(R + S) = L(R) \cup L(S)$

$L(RS) = L(R)L(S)$, $L(R^*) = L(R)^*$

$L(a) = \{a\}$

set of all possible strings over {a, b}

Example. $L((a + b)^*)$ represents the following regular language:

$$L((a + b)^*) = (L(a + b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

Is this statement true?

A language can be represented by a regular expression if and only if that language is a regular language.

Question*: can you find a regular expression for* $\{ a^n b^n \mid n \in N \}$*?*

Joke for today:

Me: The new girl in our neighborhood smiled at me today!

Wife: Be careful, she has Covid.

Me: What? How do you know?

Wife: Can’t you see she has no taste?

So, what conclusion can you draw here?

The wife has no taste either.

Regular Languages:

***Back to the Problem*:** Suppose input strings are strings over the alphabet *{a, b}* that contain exactly one substring *bb.* That is, the strings must be of the form

xbby

where ***x*** and ***y*** are strings over *{a, b}* that do not contain *bb*, ***x*** does not end in *b*, and ***y*** does not begin with *b*.

How can we describe the set of these strings formally?

Solution: let $x = (a + ba)^*$ and $y = (a + ab)^*$.

$$L((a+ba)^*) = (L(a+ba))^* = (L(a) \cup L(ba))^*$$

$$= (\{a\} \cup \{ba\})^* = \{a, ba\}^*$$

$$= \{\Lambda, a, ba, aa, aaa, \cdots, a^n, \cdots, baba \cdots, (ba)^n, \cdots, aba, \cdots\}$$

Write these down:

x could be *Φ* or *Λ*

A : a set

$A^* \equiv A^0 \cup A^1 \cup A^2 \cup A^3 \cup \cdots \cup A^n \cup \cdots$

$\equiv \emptyset \cup A^1 \cup A^2 \cup A^3 \cup \cdots \cup A^n \cup \cdots$

x : a string (could be a single-letter string)

$x^* \equiv x^0 \cup x^1 \cup x^2 \cup x^3 \cup \cdots \cup x^n \cup \cdots$

$\equiv \Lambda \cup x^1 \cup x^2 \cup x^3 \cup \cdots \cup x^n \cup \cdots$

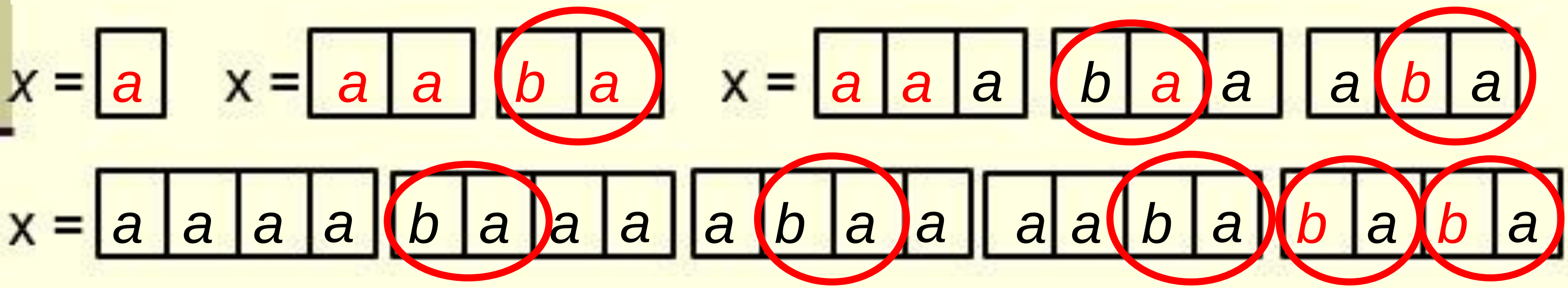
Regular Languages:

Back to the Problem: *……………………...*

……….

where *x* and *y* are strings over *{a, b}* that do not contain *bb*, *x* does not end in *b*, and *y* does not begin with *b.*

How do I know that x = (a + ba)* ?



So each x is composed of elements from {a}* or/and {ba}*.
(note that x could be equal to Λ)

Regular Languages:

Question:

x = (ba)* ?

No, because it does not cover strings like
a, aba, ababa, …..

or

aa, aaa, aaaa, …..

Regular Languages:

Question:

x = (aba)*   ?

No, why?

abaaba   *abaabaaba*

L((aba)*)={Λ, $(aba)^1$, $(aba)^2$, $(aba)^3$, …, $(aba)^n$, …}

does not cover strings like a, aa, …, ba, baa, …, ababa, abaaaba, …..

Regular Languages:

Quiz. Find a regular expression for

$$\{ ab^n \mid n \in \mathbf{N}\} \cup \{ ba^n \mid n \in \mathbf{N}\}.$$

Answer. *ab* + ba**

Quiz. Use a sentence to describe the language of

$$(b + ab)^*(\Lambda + a).$$

Answer. All strings over *{a, b}* whose substrings of *a*'s have length 1.

(or, all strings over {a, b} that do not contain the substring aa)

Regular Languages:

Quiz. Find a regular expression for

$\{ ab^n \mid n \in \mathbf{N}\} \cup \{ ba^n \mid n \in \mathbf{N}\}$.

Answer. *ab* + ba** *Why?*

L(ab + ba*) = L(ab*) + L (ba*)*

= L(a) L(b) + L(b) L(a*)*

= L(a) L(b) + L(b) L(a)**

= {a} {b} U {b} {a}**

$= \{ ab^n \mid n \in N \} \cup \{ba^n \mid n \in N \}$

Regular Languages:

Quiz. Use a sentence to describe the language of

$(b + ab)^*(\Lambda + a)$.

Answer. *All strings over {a, b} whose substrings of a's have length 1*

(*or, all strings over {a, b} that do not contain the substring aa*)

Why?

$= (b + ab)^*\Lambda + (b + ab)^*a$ ***intuitively***

$= \underline{(b + ab)^*} + \underline{(b + ab)^*a}$

b(ab)* b*(ab)*…* | *Λ*

b(ab)*b*(ab)*…* | *a*

Strings in the language of (b + ab)*(Λ + a) :

(b + ab)*(Λ + a) = (b + ab)*Λ + (b + ab)*a

(b + ab)*Λ	(b + ab)*a
b	ba
ab	aba
bab	baba
bbab	bbaba
bbabab	bbababa
.........b	.........ba

Skip the next three slides

Reg... ...ges:

How should this part be modified?

How should this part be modified?

We just proved that:

The following is an expression for all strings over *{a, b}* that do not contain the substring *aa*:

(b + ab) (Λ + a)*

Question:

What is an expression for all strings over {a, b} that do not contain the substring aaa ?

Examples : babaab | babaaba | babaabaa

Answer: *(b + ab + aab)*(Λ + a + aa)*

Regular Languages:

Known Results:

1. regular expression for strings over *{a, b*} with exactly one "*a*" ?

*b*ab**

2. exactly 2 *a*'s ?

*b*ab*ab**

exponent of b could be 0

3. exactly 3 *a*'s ?

*b*ab*ab*ab**

. . .

Regular Languages:

Is this part necessary?

Known Results:

4. regular expression for strings over {*a, b*} with even number of *a*'s ?

Answer: $(b^*ab^*ab^*)^*b^*$

$(b^*ab^*ab^*)^*b^*$

Is this part necessary?

$(b^*ab^*ab^*)^*$

covers Λ, $(b^*ab^*ab^*)$, $(b^*ab^*ab^*)^2$, $(b^*ab^*ab^*)^3$, …

but not b^*

such as *b*, *bb*, *bbb*, …

These strings also satisfy the requirement.

Regular Languages:

Known Results:

4. regular expression for strings over *{a, b*} with even number of *a*'s ?

*(b*ab*ab*)*b**   *(or, b*(b*ab*ab*))*

Question:

What is a regular expression for strings over *{a, b*} with odd number of *a*'s ?

*Would (b*ab*)* work?*

No

What is a regular expression for strings over {*a*, *b*} with odd number of *a*'s ?

*Would (b*ab*ab*)*ab* work?*

Is bab or bbab covered by this regular expression?

*Would (b*ab*ab*)*b*ab* work?*

YES

Or, b(b*ab*ab*)*ab* ?*

Algebra of Regular Expressions:

Equality: If L(R) = L(S), we say regular expressions *R* and *S* are equal and write *R* = *S*

Examples. $a + b = b + a,$ $a + a = a$

L(a+b) = {a, b} = {b, a} = L(b+a)

L(a+a) = {a, a} = {a} = L(a)

Why do we want to study regular expression algebra?

Because studying relations between regular languages directly is a very complex and big job.

Most regular languages are infinite sets. Studying relations between regular languages directly means we have to deal with union, intersection, concatenation, … of infinite sets. Big job.

On the other hand, if we instead study relations between regular expressions, we only have to deal with addition, concatenation, … , of a few representations (formulas), a much smaller job.

Algebra of Regular Expressions:

Remember: $a^* \equiv \Lambda + a + a^2 + a^3 + \cdots$

Equality: If L(R) = L(S), we say Regular expressions *R* and *S* are equal and write *R* = *S*

Examples. *aa* = a*a*   *ab ≠ ba*

L(aa*) = L(a) L(a)* = {a}{a}* = $\{a^i \mid i \in N^+\}$

= {a}*{a} = L(a)* L(a) = L(a*a)

L(ab) = L(a)L(b) = {a}{b} = {ab}

≠ {ba} = {b}{a} = L(b)L(a) = L(ba)

Algebra of Regular Expressions:

Properties of + , · , and closure

+ is commutative, associative,
  Φ is identity for +, and *R* + *R* = *R*.

$\Phi + R = R$
$\Lambda + R \neq R$

· is associative, Λ is identity for · ,
  and Φ is zero for · .

$\Lambda R = R\Lambda = R$

$\Phi R = R\Phi = \Phi$

· distributes over +

$R(S+T) = RS + RT$

Algebra of Regular Expressions:

However, RR ≠ R

Properties of

R+T = T+R

$L(R+T) = L(R) \cup L(T) = L(T) \cup L(R) = L(T+R)$

$R + \phi = \phi + R = R$

$L(R+\Phi) = L(R) \cup L(\Phi) = L(R) \cup \Phi = L(R)$

R+R = R

$L(R+R) = L(R) \cup L(R) = L(R)$

(R+S)+T = R+(S+T)

$L((R+S)+T)=L(R+S) \cup L(T)$

$= ((L(R) \cup L(S)) \cup L(T)$

$= L(R) \cup (L(S) \cup L(T))$

$= L(R+(S+T))$

Algebra of Reg[ular Express]ions:

Needed in slide 53 (number at lower right corner)

Properties of · :

$R\phi = \phi R = \phi$ $\qquad L(R\phi) = L(R)L(\phi) = L(R)\phi = \phi = L(\phi)$

$R\Lambda = \Lambda R = R$ $\qquad L(R\Lambda) = L(R)L(\Lambda) = L(R)\{\Lambda\} = L(R)$

(RS)T = R(ST) $\qquad L((RS)T) = L(RS)L(T) = (L(R)L(S))\ L(T)$

$= L(R)\ (L(S)L(T))$

$= L(R)L(ST)$

$= L(R(ST))$

Algebra of Regular Expressions:

Distributive properties:

$R(S+T) = RS + RT$

$L(R(S+T)) = L(R)L(S+T)$

$= L(R)(L(S) \cup L(T))$

$= L(R)L(S) \cup L(R)L(T)$

$= L(RS) \cup L(RT)$

$= L(RS + RT)$

Algebra of Regular Expressions:

Needed in slide 54 (number at lower right corner)

Closure Properties:

- $\emptyset^* = \Lambda^* = \Lambda$.
- $R^* = R^*R^* = (R^*)^* = R + R^*$.
- $R^* = \Lambda + R^* = \Lambda + RR^* = (\Lambda + R)^* = (\Lambda + R)R^*$.
- $R^* = (R + R^2 + \ldots + R^k)^* = \Lambda + R + R^2 + \ldots + R^{k-1} + R^k R^*$ for any $k \geq 1$.
- $R^*R = RR^*$.
- $(R + S)^* = (R^* + S^*)^* = (R^*S^*)^* = (R^*S)^*R^* = R^*(SR^*)^*$.
- $R(SR)^* = (RS)^*R$.
- $(R^*S)^* = \Lambda + (R + S)^*S$ and $(RS^*)^* = \Lambda + R(R + S)^*$.

Needed in slide 54 (number at lower right corner)

Algebra of Regular Expressions:

Closure properties:

- $\emptyset^* = \Lambda^* = \Lambda$.

Skip the next 2 slides

Proof: $\Phi^* \equiv \Phi^0 \cup \Phi^1 \cup \Phi^2 \cup \cdots$

$$= \Lambda \cup \Phi \cup \Phi \cup \cdots = \Lambda$$

$$\Lambda^* \equiv \Lambda^0 \cup \Lambda^1 \cup \Lambda^2 \cup \cdots$$

$$= \Lambda \cup \Lambda \cup \Lambda \cup \cdots = \Lambda$$

Algebra of Regular Expres[sions]

Needed in slide 54 (number at lower right corner)

Closure properties:

- $R^* = \Lambda + R^* = \Lambda + RR^* = (\Lambda + R)^* = (\Lambda + R)R^*$

Proof of $R^* = \Lambda + RR^*$.

$$L(\Lambda + RR^*) = L(\Lambda) \cup L(R)L(R)^* = \{\Lambda\} \cup L(R)^1 \cup L(R)^2 \cup \cdots$$
$$= L(R)^*.$$

Proof of $R^* = (\Lambda + R)R^*$.

$$L((\Lambda + R)R^*) = L(R^* + RR^*) = L(R)^* \cup L(R)L(R)^*$$
$$= L(R)^* \cup L(R)^+ = L(R)^*.$$

Skip the next 2 slides

Algebra of Regular Expressions:

***Explain** each inequality.*

(1). $(a + b)^* \neq a^* + b^*$. (2) $(a + b)^* \neq a^*b^*$.

***Answers**.* (1) $ab \in LHS$, but not RHS

(2) $ba \in LHS$, but not RHS

Simplify regular expression $aa(b^* + a) + a(ab^* + aa)$.

***Answer**.* $aa(b^* + a) + a(ab^* + aa)$

$= aa(b^* + a) + aa(b^* + a)$ ← • distributes over +

$= aa(b^* + a)$ ← $R + R = R$

Algebra of Regular Expressions:

Example. *Show that* $(a + aa)(a + b)^* = a(a + b)^*$.

Proof I:

$(a + aa)(a + b)^* = (a + aa)a^*(ba^*)^*$ ← $(R + S)^* = R^*(SR^*)^*$

Slides 42 and 45

$= a(\Lambda + a)a^*(ba^*)^*$ ← $R = R\Lambda$ and $\cdot$ distributes over $+$

$= aa^*(ba^*)^*$ ← $(\Lambda + R)R^* = R^*$

$= a(a + b)^*$ ← $(R + S)^* = R^*(SR^*)^*$

QED.

Skip the next 9 slides

Example. *Show that*

$(a + aa + \ldots + a^n)(a + b)^* = a(a + b)^*$ for all $n \geq 1$.

Proof (by induction): If $n = 1$, the statement becomes

$a(a + b)^* = a(a + b)^*$, obviously true.

If $n = 2$, the statement becomes

$(a + aa)(a + b)^* = a(a + b)^*$, true by a previous example (slide 53).

Assume the statement is true for $1 \leq k < n$ $(n>2)$.

We need to prove the statement is true for *n.*

Example. *Show that*

$$(a + aa + \dots + a^n)(a + b)^* = a(a + b)^* \quad \text{for all } n \geq 1.$$

Proof (conti.): The LHS of the statement for *n is*

$(a + aa + \dots + a^n)(a + b)^*$

$= a(a + b)^* + (aa + \dots + a^n)(a + b)^*$ · distributes over +

$= a(a + b)^* + a(a + aa + \dots + a^{n-1})(a + b)^*$ · distributes over +

$= a(a + b)^* + a\, a(a + b)^*$ induction assumption

$= (a + aa)(a + b)^*$ · distributes over +

$= a(a + b)^*$ induction assumption

= RHS

End of Regular Language and Finite Automata I