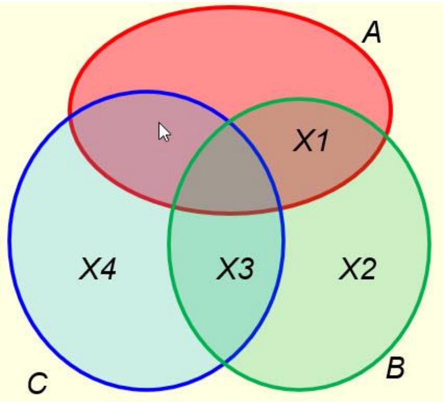
Solution to CS375 Homework Assignment 1 (40 points)

Due date: January 21, 2025

(The red boxes are text boxes. You can put your answers into the boxes directly)

1. In the following **Venn Diagram**, find an expression (in terms of set operations on the sets A, B and C; for instance, the light green region can be expressed as C-(AUB)) for the region whose four components are marked with X1, X2, X3 and X4 and put the expression in the red box below. The expression is not unique. Try to make your expression as compact as possible. (4 points)



Sol:

Expression 1: Union of components X1 and X2 = B-C, union of components X3 and X4 = C - A. So, one answer is: $(B - C) \cup (C - A)$
Expression 2: We can subtract the intersection of A and C from the union of B and C to get the specified region. So a second answer is: $(B \cup C) - (A \cap C)$

2. In the above diagram, if |A ∩ B| = 10, |A ∩ B ∩ C| = 5, |A U B U C| =150 and |X1 U X2 U X3 U X4| = 100 then |A| = ? Show your derivation in the following text box to get partial credit. (4 points)
Sol:

Note that

$$|A \cup B \cup C| - |X1 \cup X2 \cup X3 \cup X4| = |A - B| + |A \cap B \cap C|$$
$$= |A| - |A \cap B| + |A \cap B \cap C|$$

Hence, $|A|$ = 150 - 100 +10 – 5 = **55.**

3. R is a binary relation over N, the set of natural integers. (x, y) ϵ R iff *x = y (mod 7)*. R defined this way is an RST relation. List all equivalence classes of R in the following box. For each equivalence class, list all the

elements in that class. (7 points)

Sol:

There are 7 different equivalent classes: [0], [1], [2], [3], [4], [5], [6].
The contents of these equivalent classes are shown below.
[0] = {0, 7, 14, 21, …} = {7k | k ϵ N }
[1] = {1, 8, 15, 22, …} = {7k+1 | k ϵ N }
[2] = {2, 9, 16, 23, …} = {7k+2 | k ϵ N }
[3] = {3, 10, 17, 24, …} = {7k+3 | k ϵ N }
[4] = {4, 11, 18, 25, …} = {7k+4 | k ϵ N }
[5] = {5, 12, 19, 26, …} = {7k+5 | k ϵ N }
[6] = {6, 13, 20, 27, …} = {7k+6 | k ϵ N }

4. Count the number of **strings** of length 5 over *A* = {*a, b, c, d, e*} that begin with *a* and contain exactly one *b.* (5 points).

Sol:

The 2nd letter (from left to right) through the 5th letter of each string that satisfies the requirement fall into one of the following four groups:

The 2nd letter is *b*, the 3rd, the 4th and the 5th letters are *a, c, d* or *e*
The 3rd letter is *b*, the 2nd, the 4th and the 5th letters are *a, c, d* or *e*
The 4th letter is *b*, the 2nd, the 3rd and the 5th letters are *a, c, d* or *e*
The 5th letter is *b*, the 2nd, the 3rd and the 4th letters are *a, c, d* or *e*

The number of strings in each group is: 1 x 1 x 4 x 4 x 4 = 64.
So the number of strings that satisfies the requirement is **64 x 4 = 256.**

5. Use the **pigeonhole principle** to determine how many people are needed in a group to say that at least three were born on the same day of the month (each month is assumed to have 30 days). (4 points)

Sol:

At least 61 people.
Number of pigeons = number of people
Number of holes = number of days (in a month)

60 people would not work because it could be that two of them were born on each day of the month (so, 30x2, we get 60).
But if we have 61 people, and if two or less people were born on each day of the month, then we would get the conclusion that there are at most 60 people in that group, a contradiction.

6. For a set A with m elements and a set B with n elements,

(i) how many different functions $f : A \rightarrow B$ can be defined if m=4 and n=6? (2 points)

6^4=1296

(ii) if *m=4* and *n=6* then how many different *one-to-one* functions $f : A \rightarrow B$ can be defined? (4 points)

$$\frac{6!}{(6-4)!} = 360$$

7. Let $f: N_{13} \rightarrow N_{13}$ be defined by $f(x) = (6x + 5) \mod 13$. Find f^{-1} if it exists. (4 points)

Sol:

Since gcd(13, 6) = 1, **f** is bijective.
First, test values to find c such that **f**(c) = 0, e.g. **f**(10) = 0
On the other hand, from gcd(13, 6) = 1, we can get that
1 = 13×(1)+ 6×(-2). Hence, we have k = -2=11(mod 13)

Thus **f^{-1}(y) = (11y + 10) mod 13**

8. Use **induction** to prove the following equation:

$$2 + 6 + 12 + \ldots + n(n + 1) = n(n+1)(n+2)/3$$

where $n \geq 1$ (6 points).

Sol:

Let P(n) be the given equation.
For n = 1 the equation is 2 = 2. So P(1) is true.
Assume P(k) is true and prove that P(k + 1) is true.
The left side of P(k + 1) is:

2 + 6 + 12 + ... + k(k + 1) + (k + 1)(k + 2)
= k(k + 1)(k + 2)/3 + (k + 1) (k + 2) (Induction hypothesis)
= k(k + 1)(k + 2)/3 + 3(k +1)(k + 2)/3
= (k +1)(k + 2)(k + 3)/3

which is the right side of P(k + 1). So P(k + 1) is true.
So it follows from PMI that P(n) is true for all n ∈ N.